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APPLICATION OF INTEGRAL TRANSFORMS IN THE THEORY OF ELASTICITY

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DYNAMICS OF ELASTIC AND VISCOELASTIC SYSTEMS

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1. Introduction

In this course of lectures we shall deal with the theory of two-dimensional systems, both continuous and discrete. In Chapter I we present general methods of derivation of the displacement differential equations describing the considered systems and a method of their solution. We consider also, some particular cases, namely strings and beams.

The differential equation of deflection of a string or a membrane and on the other hand of a beam or a plate, have been derived on a common basis, namely the principle of virtual work implying the Hamilton principle.

Moreover, we present a unified procedure for solving the differential equations describing transverse vibrations of strings, membranes and beams and plates. This procedure concerns dynamic problems and the solutions of static problems constitute a particular case.

The above general method consists in making use of the Green function in solving the differential equations for deflections. We shall prove that a determination of the deflection of a structural element may be reduced to an integral expression containing the external loading and initial conditions, multiplied in an appropriate manner by the Green function.

Another important problem consists in the determination of the Green function. In order to unify this procedure we consistently apply integral and finite transforms.

In dynamic problems we first use the integral Laplace transform with respect to the time t, in order to eliminate the time from the differential equation for deflection. Next, we make use of the Fourier transform or a finite transform, depending on the prescribed boundary conditions. Thus, for a plate strip simply supported on its boundaries, we first apply the exponential Fourier transform and then a finite sine transform. The inversion of the integral transforms leads to the Green function. We arrive at the final results by substituting the Green function into the integral expression examined in Sec. 4. The general procedure presented in this Chapter can be extended to more complicated systems, e.g. shells and to discrete gridworks. Finally we demonstrate an application of an analogous method to the problem of free and forced vibrations of systems the material of which is viscoelastic.

2. The principle of virtual work and Hamilton's principle

Consider an elastic body subject to the action of external forces; the latter include body forces and surface tractions. We assume that the external loadings depend on position x and time t. These sources produce in the body a displacement field $\underset{\sim}{u}(\underset{\sim}{x},t)$ and the associated with this field state of strain ε_{ij} and stress σ_{ij}.

In linear elasticity we define the strain tensor as follows:

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad i,j = 1,2,3. \tag{2.1}$$

The components of the state of stress are linear functions of strain. The generalized Hooke law has the form

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}. \tag{2.2}$$

The quantities μ,λ are material constants called the Lamé constants. The above equations are completed by the equations of motion which are derived from the fundamental principles of mechanics, namely the principle of conservation of linear and angular momenta. They have the form

$$\sigma_{ji,j} + X_i = \rho\ddot{u}_i, \quad \sigma_{ji} = \sigma_{ij}, \quad \underset{\sim}{x} \in V, \quad t > 0. \tag{2.3}$$

where $\underset{\sim}{X}$ is the vector of the body forces, referred to a unit volume, ρ is the density and $\ddot{u}_i = \dfrac{\partial^2 u_i}{\partial t^2}$ the acceleration.

Equations (2.1) - (2.3) constitute the system of equations of linear elasticity. They should be completed by the boundary and initial conditions. Assume that the surface A bounding the body consists of two parts, $A = A_u + A_\sigma$. On A_u there are prescribed displacements while on A_σ tractions. Thus, we have the boundary conditions

$$\begin{aligned} u_i(\underset{\sim}{x},t) &= \hat{u}_i(\underset{\sim}{x},t), \quad \underset{\sim}{x} \in A_u, \quad t > 0, \\ \sigma_{ji}(\underset{\sim}{x},t)n_j(\underset{\sim}{x}) &= \hat{p}_i(\underset{\sim}{x},t), \quad \underset{\sim}{x} \in A_\sigma, \quad t > 0, \end{aligned} \tag{2.4}$$

where $\hat{u}_i$ and $\hat{p}_i$ are known functions.

The initial conditions have the form

$$u_i(\underset{\sim}{x},0) = f_i(\underset{\sim}{x}), \ \dot{u}_i(\underset{\sim}{x},0) = g_i(\underset{\sim}{x}), \ \underset{\sim}{x} \in V, \ t = 0. \tag{2.5}$$

They express the fact that at the initial instant $t = 0$ the distribution of the displacement field $f_i(\underset{\sim}{x})$ and its velocity $g_i(\underset{\sim}{x})$ are prescribed.

The principle of virtual work and Hamilton's principle are of a fundamental importance in deriving the differential equations for the vibrations of strings, beams, membranes, plates and shells.

The principle of virtual work has the form

$$\int_V (X_i - \rho\ddot{u}_i)\delta u_i \mathrm{d}V + \int_{A_\sigma} p_i \delta u_i \mathrm{d}A = \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{d}V. \tag{2.6}$$

Here δu_i is the virtual increment of the displacement, $\delta\varepsilon_{ij}$ the virtual increment of strain. We assume that the above increments are arbitrary and sufficiently smooth (of class $C^{(2)}$) and that they satisfy the kinematic conditions on the surface A. We require that the virtual increments δu_i vanish on the surface A_u and are arbitrary on A_σ.

The principle of virtual work states that the sum of the virtual work performed by the body forces, inertia forces and surface forces in arbitrary virtual displacements is equal to the virtual work of the internal forces.

Introducing the concept of the work of strain

$$\mathcal{W}_\varepsilon = \int_V W_\varepsilon \mathrm{d}V = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\mathrm{d}V = \int_V (\mu\varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\varepsilon_{kk}\varepsilon_{mm})\mathrm{d}V, \tag{2.7}$$

we write Equation (2.6) in the form

$$\int_V (X_i - \rho\ddot{u}_i)\delta u_i \mathrm{d}V + \int_{A_\sigma} p_i \delta u_i \mathrm{d}A = \delta\mathcal{W}_\varepsilon. \tag{2.8}$$

The integrand in the expression for the work of strain is a positive definite quadratic form. The necessary and sufficient condition that the integrand in (2.7) be of such a form is the following:

$$3\lambda + 2\mu > 0, \quad \mu > 0. \tag{2.9}$$

Observe that for a static problem all causes and resulting displacements depend on position, i.e. on $\underset{\sim}{x}$. Equation (2.8) takes the form

$$\int_V X_i \delta u_i \mathrm{d}V + \int_{A_\sigma} p_i \delta u_i \mathrm{d}A = \delta \mathcal{W}_\varepsilon . \tag{2.9'}$$

On the basis of the principle of virtual work we can (by varying the state of displacement) derive a very general minimum principle for a non-stationary displacement field.

Let us consider an elastic body continuously changing its state between the instants $t = t_1$ and $t = t_2$. Let us compare the true displacements taking place in the body with the displacements $u_i + \delta u_i$, the variations δu_i being chosen such that they vanish at the instants $t = t_1$ and $t = t_2$:

$$\delta u_i(\underset{\sim}{x}, t_1) = 0, \quad \delta u_i(\underset{\sim}{x}, t_2) = 0. \tag{2.10}$$

If we integrate equation (2.9) over t from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2} \delta \mathcal{W}_\varepsilon \mathrm{d}t = \int_{t_1}^{t_2} \delta \mathcal{L} \mathrm{d}t - \rho \int_{t_1}^{t_2} \mathrm{d}t \int_V \ddot{u}_i \delta u_i \mathrm{d}V \tag{2.11}$$

where

$$\delta \mathcal{L} = \int_V X_i \delta u_i \mathrm{d}V + \int_{A_\sigma} p_i \delta u_i \mathrm{d}A .$$

The variation of the kinetic energy is given by the formula

$$\delta \mathcal{K} = \int_V \rho \dot{u}_i \delta \dot{u}_i \mathrm{d}V = \int_V \rho \frac{\partial}{\partial t}(\dot{u}_i \delta u_i) \mathrm{d}V - \int_V \rho \ddot{u}_i \delta u_i \mathrm{d}V$$

since

$$\mathcal{K} = \frac{\rho}{2} \int_V \dot{u}_i \dot{u}_i \mathrm{d}V .$$

Integrating $\delta \mathcal{K}$ from t_1 to t_2, and bearing in mind the assumption (2.10) we obtain

$$\int_{t_1}^{t_2} \delta\mathcal{K}\,dt = -\rho\int_{t_1}^{t_2} dt\int_V \ddot{u}_i\,\delta u_i\,dV. \tag{2.12}$$

Substituting from (2.12) into (2.11), we have

$$\delta\int_{t_1}^{t_2} (\mathcal{W}_\varepsilon - \mathcal{K})dt = \int_{t_1}^{t_2} \delta\mathcal{L}\,dt. \tag{2.13}$$

If the external forces are conservative they possess a potential and in this case

$$\delta\int_{t_1}^{t_2} (\mathcal{W}_\varepsilon - \mathcal{K} - \mathcal{L})dt = 0 \tag{2.14}$$

Denoting by $\pi = \mathcal{W}_\varepsilon - \mathcal{L}$ the total potential energy of the system we present the Hamilton principle in the final form

$$\delta\int_{t_1}^{t_2} (\pi - \mathcal{K})dt = 0 \tag{2.15}$$

It states therefore that the integral (2.15) takes an extremum value.

3. Transverse vibrations of simple one- and two-dimensional systems.

In this Section, on the basis of the principle of virtual work and the Hamilton principle we shall derive the differential equations for vibrations of a string and a membrane and the equation describing the vibrations of a beam and a plate. We shall emphasize here the evident analogies in the derivations.

(*a*) Consider a string in tension along the x_1-axis between the points A and B. The constant tension in the string is denoted by S and its length by l. Assume that in the xz-plane a load $q(x,t)$ acts per unit length of the string. This loading produces a deflection of the string $w(x,t)$ in the xz-plane. We have assumed here that in the cross-section of the string there occurs a homogeneous state of stress $\sigma_{xx} = \frac{S}{A}$ and that the deflection of the string is independent of y.

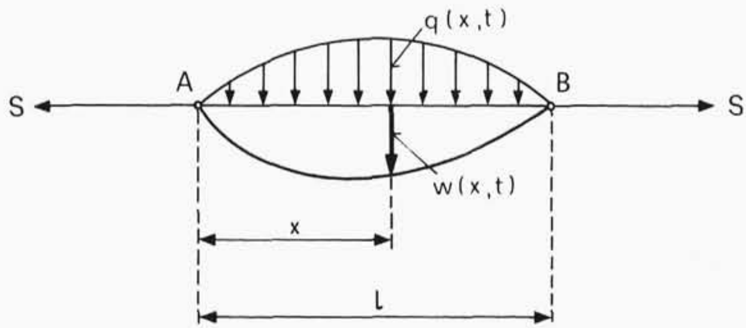

Fig. 3.1

In deriving the differential equation for the deflection of the string we should remember that the derived equation is approximate, in view of the simplifying assumptions made above.

To derive the differential equation for the deflection of the string we have made use of the principle of virtual work (2.8) of Section 2. Neglecting the influence of the weight of the string ($X_i = 0$) on its deflection, we write Equation (2.8) of Section 2 in the form

$$-\int_0^l \rho\ddot{w}\delta w A dx + \int_0^l q\delta w dx = \delta\mathcal{W}_\varepsilon, \quad A = \int_A dydz. \tag{3.1}$$

The quantity $\mathcal{W}_\varepsilon$ is obtained on the basis of the following considerations. Under the influence of the external loading the length dx of a linear element of the string undergoes an extension. The work of deformation is the following:

$$\mathcal{W}_\varepsilon = S\int_0^l (\mathrm{d}s-\mathrm{d}x) = S(l'-l). \tag{3.2}$$

We have denoted by $\mathrm{d}s$ the length of the element $\mathrm{d}x$ after the deformation. Summing the above deformations we obtain the quantity $\mathcal{W}_\varepsilon = S(l'-l)$. The absence of the coefficient $\frac{1}{2}$ in the right-hand side of Equation (3.2) is due to the fact that at the instant of application of the loading the tension S already had its final value. Strictly speaking we should write (3.2) in the form $\mathcal{W}_\varepsilon = (S+\mathrm{d}S)(l'-l)$ where $\mathrm{d}S$ is the increment of tension S due to the loading q. However, this increment is very small as compared with S. Taking into account that

$$\mathcal{W}_\varepsilon = S\int_0^l \{[1 + (\frac{\partial w}{\partial x})^2]^{\frac{1}{2}} - 1\}\mathrm{d}x, \tag{3.3}$$

and expanding the expression $\left[1 + (\frac{\partial w}{\partial x})^2\right]^{\frac{1}{2}}$ in series we arrive at the formula

$$\mathcal{W}_\varepsilon = \frac{S}{2}\int_0^l (\frac{\partial w}{\partial x})^2 \mathrm{d}x \tag{3.4}$$

Since $(\frac{\partial w}{\partial x})^2 \ll 1$ we have retained only the first two terms in the expansion of the function $\left[1 + (\frac{\partial w}{\partial x})^2\right]^{\frac{1}{2}}$. Performing the variation of the work of deformation

$$\delta\mathcal{W}_\varepsilon = S\int_0^l \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\mathrm{d}x = S\left[\frac{\partial w}{\partial x}\delta w\right]_0^l - S\int_0^l \frac{\partial^2 w}{\partial x^2}\delta w\mathrm{d}x,$$

we represent Equation (3.1) in the form

$$\int_0^l (S\frac{\partial^2 w}{\partial x^2} - \rho A\frac{\partial^2 w}{\partial t^2} + q)\delta w\mathrm{d}x = S\left[\frac{\partial w}{\partial x}\delta w\right]_0^l. \tag{3.5}$$

If the string is clamped at its ends $x = 0$ and $x = l$ we have

$$w(0,t) = 0, \quad w(l,t) = 0. \tag{3.6}$$

In deriving the principle of virtual work we assumed that when the displacements are prescribed, then $\delta u_i = 0$. In our case we have prescribed displacements (3.6) at the ends of string and therefore at these points we have $\delta w(0,t) = 0$, $\delta w(l,t) = 0$. Consequently, the right hadn side of Equation (3.5) is zero. In view of the arbitrariness of the virtual displacement δw, the left-hand side of the homogeneous equation leads to the differential equation

$$c^2 \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial t^2} + \frac{1}{\sigma} q(x,t) = 0, \tag{3.7}$$

$$0 < x < l, \quad t > 0,$$

where we have introduced the notation

$$c^2 = {}^S/\rho A, \qquad \sigma = \rho A.$$

The differential equation for the transverse vibrations of the string (3.7) should be completed by the boundary conditions and the initial conditions

$$w(x,0) = f(x), \quad \dot{w}(x,0) = g(x), \quad 0 < x < l, \quad t = 0. \tag{3.8}$$

(*b*) Consider now transverse vibrations of a membrane. By a membrane we understand a plate whose thickness is very small compared with its other linear dimensions. A membrane offers no resistance to bending. It constitutes the two-dimensional counterpart of a string.

Consider a membrane in a homogeneous tension S in the plane x_1x_2, with contour c. Assume that normal to the plane x_1x_2 there acts the loading $q(x_1,x_2,t)$. Under the influence of the tension S and the loading q there arises in the membrane a two-dimensional state of stress (described by the normal stresses $\sigma_{11} = \sigma_{22}$ homogeneously distributed over the thickness of the membrane); the membrane then undergoes a deflection in the direction of the x_3-axis, denoted here by $w(x_1,x_2,t)$.

Let us derive the equation of deflection of the membrane, on the basis of the principle of virtual work, by varying the displacement

Equation (2.8):

$$-\iint_A h\rho\ddot{w}\delta w \mathrm{d}A + \iint_A q\delta w \mathrm{d}A = \delta \mathcal{W}_\varepsilon . \qquad (3.9)$$

We have neglected here the influence of the weight of the membrane ($X_i = 0$) on its deflection. Under the influence of the external loading an arbitrary surface element A_0 of the membrane undergoes a deflection. Separating this element and subjecting it to the tension S we find that its surface increases by the value of the integral $\iint S\mathrm{d}u_n \mathrm{d}s$, where $\mathrm{d}s$ is an element of arc of the contour c_0 and $\mathrm{d}u_n$ is the displacement in the direction normal to the curve c_0. The increment in the surface is shaded in Figure 3.2. Taking into account that for every surface element the work of deformation is the product of the tension S and the increment of the surface, we have for the whole membrane

$$\mathcal{W}_\varepsilon = S(A'-A), \qquad (3.10)$$

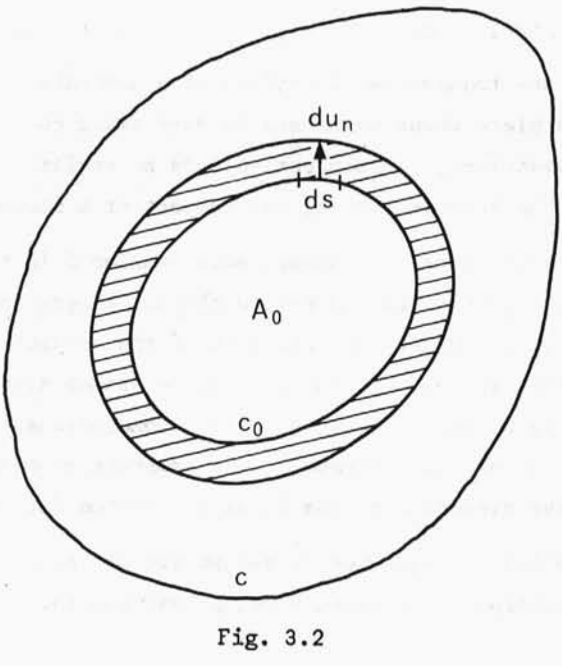

Fig. 3.2

where A' is the area of the surface of the deformed membrane. Thus, Equation (3.10) constitutes a complete counterpart of Equation (3.2).

In the expression for the work of deformation (3.10) S is the initial tension. It changes insignificantly due to the action of the external loading $q(x_1,x_2,t)$; this increment is very small indeed as compared with the initial tension and can be neglected in the expression (3.10).

It is known from differential geometry that the change in area of the surface is given by the formula

$$A'-A = \iint_A \{\left[1 + (\frac{\partial w}{\partial x_1})^2 + (\frac{\partial w}{\partial x_2})^2\right]^{\frac{1}{2}} - 1\}dx_1 dx_2. \quad (3.11)$$

Expanding the integrand of (3.11) in series and confining ourselves to small deflections we obtain

$$\mathcal{W}_\varepsilon = \frac{S}{2} \iint_A \left[(\frac{\partial w}{\partial x_1})^2 + (\frac{\partial w}{\partial x_2})^2\right]dA, \quad dA = dx_1 dx_2. \quad (3.12)$$

Let us now determine the variation of the work of deformation. We have

$$\delta \mathcal{W}_\varepsilon = S\iint_A w_{,\alpha}\delta w_{,\alpha}dA, \quad \alpha = 1,2. \quad (3.13)$$

The integral appearing in the right-hand side of the expression (3.13) can be transformed as follows:

$$\delta \mathcal{W}_\varepsilon = S\iint_A \left[(w_{,\alpha}\delta w)_{,\alpha} - w_{,\alpha\alpha}\delta w\right]dA \quad (3.14)$$

Making use of the Green transformation in the plane we reduce (3.14) to the form

$$\delta \mathcal{W}_\varepsilon = S \int_C \frac{\partial w}{\partial n}\delta w ds - S\iint_A w_{,\alpha\alpha}\delta w dA, \quad (3.15)$$

where $\frac{\partial w}{\partial n}$ denotes the derivative of the deflection along the normal to the boundary c. Introducing (3.15) into (3.9) we arrive at the equation

$$\iint_A (Sw_{,\alpha\alpha}-\sigma h\ddot{w}+q)\delta w dA - \int_c \frac{\partial w}{\partial n}\delta w ds = 0, \tag{3.16}$$

where $\sigma = \rho h$ is the mass per unit surface of the membrane. If on the boundary c the displacement $w(s)$ is prescribed, then $\delta w = 0$ on c. There remains in (3.16) the first integral only. In view of the assumed arbitrariness of the displacement δw within the membrane (3.16) leads to the differential equation

$$S\nabla^2 w - \sigma\frac{\partial^2 w}{\partial t^2} + q = 0, \quad \underset{\sim}{x} \in A, \; t > 0, \; \underset{\sim}{x} = (x_1,x_2), \tag{3.17'}$$

or

$$c^2\nabla^2 w-\ddot{w} = -q/\sigma, \qquad c^2 = {}^S/\sigma. \tag{3.17''}$$

This is the differential equation for transverse vibrations of a membrane. It should be completed by the boundary condition

$$w(s,t) = 0, \quad s \in c, \; t > 0, \tag{3.18}$$

and the initial conditions

$$w(\underset{\sim}{x},0) = f(\underset{\sim}{x}), \quad \dot{w}(\underset{\sim}{x},0) = g(\underset{\sim}{x}), \quad \underset{\sim}{x} \in A, \; t = 0. \tag{3.19}$$

(c) *The differential equation of the transverse vibrations of a rod.*

We proceed to derive the differential equation governing the transverse vibrations of a rod, on the basis of Hamilton's principle. We calculate the work of deformation $\mathcal{W}_\varepsilon$, the kinetic energy $\mathcal{K}$ and the variation of work of the external forces $\delta\mathcal{L}$, i.e. the quantities which enter into the variational expression (2.13).

The work of deformation has the form

$$\mathcal{W}_\varepsilon = \tfrac{1}{2}\int\limits_V \sigma_{xx}\varepsilon_{xx}\,dV. \tag{3.20}$$

We neglect in the expression for $\mathcal{W}_\varepsilon$ the influence of the transverse forces; it is very small for beams used in civil engineering structures, the longitudinal dimensions of which are considerably greater than the transverse ones. Taking into account that

$$\sigma_{xx} = E\varepsilon_{xx}, \qquad \varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}, \tag{3.21}$$

where w is the deflection of the rod, and integrating over the length and cross-section of the rod, we obtain

$$\mathcal{W}_\varepsilon = \frac{E}{2}\int\limits_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx \iint\limits_A z^2\,dy\,dz, = \frac{EI}{2}\int\limits_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx. \tag{3.22}$$

I is the moment of inertia of the cross-section. (Cf. Fig. 3.3).

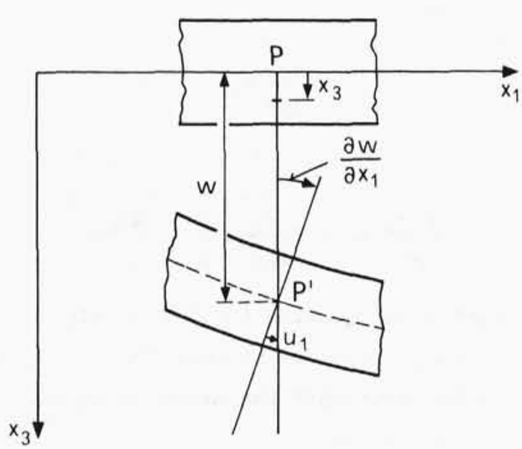

Fig. 3.3

The kinetic energy of the translational motion has the form

$$\mathcal{K} = \tfrac{1}{2}\rho\int_V (\dot{w})^2 dV = \tfrac{1}{2}\sigma\int_0^l (\dot{w})^2 dx, \quad \sigma = \rho A. \tag{3.23}$$

Denoting by q the loading per unit length of the rod, we have

$$\delta\mathcal{L} = \int_0^l q\,\delta w\,dx. \tag{3.24}$$

Hamilton's principle has the form

$$\delta\int_{t_1}^{t_2} dt\int_0^l dx\left[\frac{EI}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 - \tfrac{1}{2}\sigma(\dot{w})^2\right] = \int_{t_1}^{t_2} dt\int_0^l q\,\delta w\,dx. \tag{3.25}$$

Let us calculate the variations

$$\delta\int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = 2\int_0^l \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 \delta w}{\partial x^2}dx.$$

Taking into account the identity

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} - \frac{\partial^4 w}{\partial x^4}\delta w = \frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta w}{\partial x} - \frac{\partial^3 w}{\partial x^3}\delta w\right)$$

we obtain

$$\delta\int_0^l \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx = 2\int_0^l \frac{\partial^4 w}{\partial x^4}\delta w\,dx + 2\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta w}{\partial x} - \frac{\partial^3 w}{\partial x^3}\delta w\right]_0^l. \tag{3.26}$$

Examine the second integral in Equation (3.25). Integrating by parts with respect to t and taking into account that for $t = t_1$ and $t = t_2$ we have $\delta w = 0$, then, in accordance with the assumptions made in deriving the Hamilton's principle, we obtain

$$\frac{\sigma}{2}\delta\int_{t_1}^{t_2} dt\int_0^l (\dot{w})^2 dx = -\sigma\int_{t_1}^{t_2} dt\int_0^l \ddot{w}\,\delta w\,dx. \tag{3.27}$$

Introducing the above results in (3.25) we arrive at the relation

$$\int_{t_1}^{t_2} dt \int_0^l \left(EI\frac{\partial^4 w}{\partial x^4} + \sigma\frac{\partial^2 w}{\partial t^2} - q\right)\delta w dx + EI\int_{t_1}^{t_2} dt \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta w}{\partial x} - \frac{\partial^3 w}{\partial x^3}\delta w\right]_0^l = 0. \qquad (3.28)$$

The second term vanishes as a result of the boundary conditions. If the rod is simply supported at the cross section $x = 0, l$ we have

$$w = 0, \quad M = -EI\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{and also } \delta w = 0.$$

If the end is fixed, then

$$w = 0, \ \frac{\partial w}{\partial x} = 0 \text{ and also } \delta w = 0, \ \frac{\partial \delta w}{\partial x} = 0.$$

Finally, if the rod is free at the end, we have the conditions

$$M = -EI\frac{\partial^2 w}{\partial x^2} = 0, \quad T = -EI\frac{\partial^3 w}{\partial x^3} = 0.$$

Taking into account the above boundary conditions, we have from (3.28)

$$\int_{t_1}^{t_2} dt \int_0^l \left(EI\frac{\partial^4 w}{\partial x^4} + \sigma\frac{\partial^2 w}{\partial t^2} - q\right)\delta w dx = 0.$$

Since this relation has to be satisfied for every value of δw and $t(t_1<t<t_2)$, we obtain the differential equation of transverse vibration of a rod

$$EI\frac{\partial^4 w}{\partial x^4} + \sigma\frac{\partial^2 w}{\partial t^2} - q = 0, \qquad (3.29')$$

or

$$c^2\frac{\partial^4 w}{\partial x^4} + \ddot{w} = q/_\sigma, \quad c^2 = \frac{EI}{\sigma}, \ \sigma = \rho A. \qquad (3.29'')$$

Equation (3.29') is associated with the two initial conditions

$$w(x,0) = f(x), \ \dot{w}(x,0) = g(x), \ 0 < x < l, \quad t = 0. \qquad (3.30)$$

(*d*) *The differential equation of the transverse vibrations of a thin plate.*

Considering the deformation of a thin plate, i.e. assuming its thickness to be small in comparison with other dimensions, we make the following simplifying assumptions:

a) points lying on a normal to the middle surface remain on the normal to the middle surface after deformation;

b) during the deformation no strains are induced on the middle surface;

c) the influence of the shearing stresses σ_{31}, σ_{32} on the deformation of the plate is neglected.

The displacements $u_\beta (\beta = 1,2)$ are proportional to the angle

$$u_\beta = -x_3 w_{,\beta}. \tag{3.31}$$

The strains are then

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) = -x_3 w_{,\alpha\beta} \tag{3.32}$$

Using the formulae for plane stress

$$\sigma_{\alpha\beta} = 2G(\varepsilon_{\alpha\beta} + \frac{\nu}{1-\nu}\delta_{\alpha\beta}\varepsilon_{kk}), \quad \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22}, \tag{3.33}$$

we obtain

$$\sigma_{\alpha\beta} = -\frac{2Gx_3}{1-\nu}\left[(1-\nu)w_{,\alpha\beta} + \nu\delta_{\alpha\beta}w_{,kk}\right]. \tag{3.34}$$

We introduce now the resultants of the stresses acting in the plate: the bending moments are given by

$$M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta}x_3 dx_3, \qquad \alpha,\beta = 1,2. \tag{3.35}$$

Performing the integration, we have

$$M_{\alpha\beta} = -N\left[(1-\nu)w_{,\alpha\beta} + \delta_{\alpha\beta}w_{,kk}\right]. \tag{3.36}$$

where

$$N = \frac{Eh^3}{12(1-\nu^2)}$$

is the flexural rigidity of the plate.

Let us calculate the strain energy of the plate. Neglecting the shearing stresses σ_{13}, σ_{23} and the normal stress σ_{33} we obtain the expression

$$\mathcal{W} = \tfrac{1}{2} \int_V \sigma_{\alpha\beta}\varepsilon_{\alpha\beta} dV, \quad \alpha,\beta = 1,2, \tag{3.37'}$$

i.e.

$$\mathcal{W} = \frac{1}{2E} \int_V \left[(1+\nu)\sigma_{\alpha\beta}\sigma_{\alpha\beta} - \nu\sigma_{kk}\sigma_{nn}\right] dV; \tag{3.37''}$$

where the integration is performed over the entire volume of the plate. Replacing the stresses by the displacement w (Equation (3.34)) and performing the integration with respect to $x_3 (dV = dA dx_3)$, we obtain

$$\mathcal{W} = \frac{N}{2} \iint_A \left[(1-\nu)w_{,\alpha\beta}w_{,\alpha\beta} + \nu(w_{,\alpha\alpha})^2\right] dA. \tag{3.38}$$

The differential equation of the transverse vibrations of the plate can be derived from Hamilton's principle

$$\delta \int_{t_1}^{t_2} (\mathcal{W} - \mathcal{K}) dt = \int_{t_1}^{t_2} \delta\mathcal{L}\, dt. \tag{3.39}$$

The strain energy is expressed by Equation (3.38), the kinetic energy by the formula

$$\mathcal{K} = \tfrac{1}{2}\sigma \iint_A (\dot{w})^2 dA, \; \sigma = \rho h, \tag{3.40}$$

and the variation of the work done by external forces has the form

$$\delta \mathcal{L} = \iint_A q \delta w \mathrm{d}A. \tag{3.41}$$

Here q denotes the load acting on the plate.

Let us perform the variation of $\delta \mathcal{W}$:

$$\delta \mathcal{W} = \delta \mathcal{W}' + \delta \mathcal{W}'' = N \iint_A \nabla^2 w \nabla^2 (\delta w) \mathrm{d}A$$
$$+ N(1-\nu) \iint_A \left[2 \frac{\partial^2 w}{\partial x_1 \partial x_2} \delta \left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right) - \frac{\partial^2 w}{\partial x_1^2} \delta \left(\frac{\partial^2 w}{\partial x_2^2} \right) - \frac{\partial^2 w}{\partial x_2^2} \delta \left(\frac{\partial^2 w}{\partial x_1^2} \right) \right] \mathrm{d}A. \tag{3.42}$$

Now, applying the two-dimensional Green's identity we obtain

$$\delta \mathcal{W}' = N \iint_A \nabla^2 w \nabla^2 (\delta w) \mathrm{d}A = N \left\{ \iint_A \nabla^4 w \delta w \mathrm{d}A + \int_c \left(\nabla^2 w \frac{\partial \delta w}{\partial n} - \delta w \frac{\partial \nabla^2 w}{\partial n} \right) \mathrm{d}s \right\}. \tag{3.43}$$

The symbol ∇^4 denotes

$$\nabla^4 = \frac{\partial^4}{\partial x_1^4} + 2 \frac{\partial^4}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4}{\partial x_2^4}.$$

The integral $\delta \mathcal{W}''$ can be written in the form

$$\delta \mathcal{W}'' = -N(1-\nu) \iint_A \left(\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} \right) \mathrm{d}A = -N(1-\nu) \int_c (q_1 \cos \vartheta + q_2 \sin \vartheta) \mathrm{d}s, \tag{3.44}$$

where

$$q_1 = \frac{\partial \delta w}{\partial x_1} \frac{\partial^2 w}{\partial x_2^2} - \frac{\partial \delta w}{\partial x_2} \frac{\partial^2 w}{\partial x_1 \partial x_2}, \quad q_2 = \frac{\partial \delta w}{\partial x_2} \frac{\partial^2 w}{\partial x_1^2} - \frac{\partial \delta w}{\partial x_1} \frac{\partial^2 w}{\partial x_1 \partial x_2}. \tag{3.45}$$

Let us transform the quantities $\frac{\partial \delta w}{\partial x_1}$, $\frac{\partial \delta w}{\partial x_2}$ appearing in Equations (9.45)

$$\frac{\partial \delta w}{\partial x_1} = \frac{\partial \delta w}{\partial n} \cos \vartheta - \frac{\partial \delta w}{\partial s} \sin \vartheta, \quad \frac{\partial \delta w}{\partial x_1} = \frac{\partial \delta w}{\partial n} \sin \vartheta + \frac{\partial \delta w}{\partial s} \cos \vartheta. \tag{3.46}$$

Inserting now (3.45), (3.46) and (3.44) we obtain

$$\delta W'' = -N(1-\nu)\int_V \frac{\partial \delta w}{\partial n}\left[\frac{\partial^2 w}{\partial x_2^2}\cos^2\vartheta + \frac{\partial^2 w}{\partial x_1^2}\sin^2\vartheta - 2\frac{\partial^2 w}{\partial x_1 \partial x_2}\cos\vartheta\sin\vartheta\right]ds \tag{3.47}$$

$$-N(1-\nu)\int_c \frac{\partial \delta w}{\partial s}\left[\left(\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2}\right)\cos\vartheta\sin\vartheta + \frac{\partial^2 w}{\partial x_1 \partial x_2}(\sin^2\vartheta - \cos^2\vartheta)\right]ds.$$

The second integral in Equation (3.47) can be integrated by parts to give

$$N(1-\nu)\int_c \frac{\partial \delta w}{\partial s} f(x_1,x_2)\,ds = N(1-\nu)\left\{\left| f(x_1,x_2)\,\delta w\right|_c - \int_c \delta w\frac{\partial f}{\partial s}ds\right\}. \tag{3.48}$$

Since δw vanishes on the boundary of the plate only the second term of the right side of Equation (3.48) remains.

The variation of the kinetic energy has the form

$$\delta\int_{t_1}^{t_2} \mathcal{K}\,dt = -\sigma\int_{t_1}^{t_2} dt\iint_A \ddot{w}\,\delta w\,dA. \tag{3.49}$$

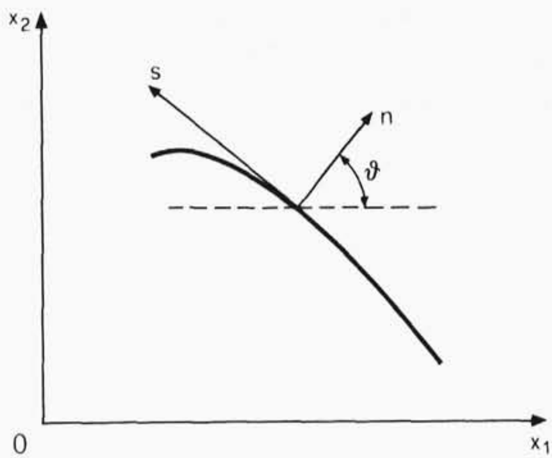

Fig. 3.4

Thus, all the expressions occurring in Equation (3.39) are now known. Hamilton's principle thus takes the form

$$\int_{t_1}^{t_2} \mathrm{d}t \left\{ \iint_A \left[N\nabla^4 w + \sigma\ddot{w} - q\right] \delta w \mathrm{d}A \right. \tag{3.50}$$

$$+ N\int_C \left[\nabla^2 w - (1-\nu)\left(\frac{\partial^2 w}{\partial x_1^2}\sin^2\vartheta + \frac{\partial^2 w}{\partial x_2^2}\cos^2\vartheta - 2\frac{\partial^2 w}{\partial x_1 \partial x_2}\sin\vartheta\cos\vartheta\right)\right]\frac{\partial \delta w}{\partial n}\mathrm{d}s$$

$$- N\int_C \left[\frac{\partial \nabla^2 w}{\partial n} - (1-\nu)\frac{\partial}{\partial s}\left(\left(\frac{\partial^2 w}{\partial x_1^2} - \frac{\partial^2 w}{\partial x_2^2}\right)\cos\vartheta\sin\vartheta + \frac{\partial^2 w}{\partial x_1 \partial x_2}(\sin^2\vartheta - \cos^2\vartheta)\right)\right]\delta w \mathrm{d}s = 0.$$

We shall now prove that the integral over C the boundary of the plate vanishes for homogeneous boundary conditions. Consider the curvilinear contour of the plate C. The resultant of the stress, the bending moment M_{nn} and the torsion M_{ns} can be expressed in terms of the moments M_{11}, M_{22}, M_{12} as follows:

$$M_{nn} = M_{11}\cos^2\vartheta + M_{22}\sin^2\vartheta + M_{12}\sin 2\vartheta,$$

$$M_{ns} = \tfrac{1}{2}(M_{22} - M_{11})\sin 2\vartheta + M_{12}\cos 2\vartheta. \tag{3.51}$$

Let us now introduce an invariant obtained by contraction of the relations (3.36)

$$M_{11} + M_{22} = -(1+\nu)N\nabla^2 w. \tag{3.52}$$

The transverse force is given by the formulae

$$Q_1 = -N\frac{\partial \nabla^2 w}{\partial x_1}, \qquad Q_2 = -N\frac{\partial \nabla^2 w}{\partial x_2}. \tag{3.53}$$

The transverse force on the boundary C has the form

$$Q_n = Q_1\cos\vartheta + Q_2\sin\vartheta = -N\frac{\partial \nabla^2 w}{\partial n}.$$

In view of (3.51), (3.52), (3.54) we represent Equation (3.50) in the form

$$\int_{t_1}^{t_2} \mathrm{d}t \iint_A (N\nabla^4 w + \sigma\ddot{w} - q)\,\delta w\,\mathrm{d}A = \int_C \left[M_{nn}(w)\frac{\partial \delta w}{\partial n} - V_n(w)\,\delta w\right]\mathrm{d}s. \tag{3.55}$$

where $V_n(w) = Q_n(w) + \dfrac{\partial M_{ns}(w)}{\partial s}$. The curvilinear integral on the right hand side of Equation (3.55) vanishes, since all the boundary conditions are satisfied. If the boundary is simply supported over its contour C, then $M_{nn} = 0$, $w = 0$ and therefore $\delta w = 0$. If the boundary is clamped, then $w = 0$, $\dfrac{\partial w}{\partial n} = 0$ on C and hence $\delta w = 0$ and $\dfrac{\partial \delta w}{\partial n} = 0$. Finally, in the case of a boundary free from tractions: $M_{nn}(w) = 0$ and $V_n(w) = 0$. The quantity $V_n(w)$ is the sum of the transverse forces acting on the boundary (the so-called Kelvin-Tait boundary condition).

Equation (3.55) takes the form

$$\int_{t_1}^{t_2} \mathrm{d}t \iint_A (N\nabla^4 w + \sigma\ddot{w} - q)\,\delta w\,\mathrm{d}A = 0. \tag{3.56}$$

In view of the arbitrariness of the virtual displacement δw the bracket in the integrand should vanish and this equation holds for every instant t where $t_1 < t < t_2$.

Thus, the differential equation for the transverse vibrations of a plate takes the form

$$c^2\nabla^4 w + \ddot{w} = \frac{1}{\sigma}q(x_1,x_2,t), \quad (x_1,x_2) \in A,\ t > 0, \tag{3.57}$$

where

$$c^2 = N/_{\sigma}, \qquad \sigma = \rho h.$$

The differential equation (3.57) should be completed by the initial conditions

$$w(x_1,x_2,0) = f(x_1,x_2), \quad \dot{w}(x_1,x_2,0) = g(x_1,x_2),\ \underset{\sim}{x} \in A,\ t = 0. \tag{3.58}$$

Knowing the deflection surface w, we can determine the bending and torsional moments from the formulae (3.36).

4. General solution of differential equations of transverse vibrations.

In the preceding section, on the basis of the principle of virtual work and Hamilton's principle we derived the following differential equations:

$$c^2\frac{\partial^2 w}{x} - \ddot{w} = -\frac{1}{\sigma}q(x,t), \qquad c^2 = S/_\sigma, \quad \sigma = A\rho, \tag{4.1}$$

$$c^2\nabla^2 w - \ddot{w} = -\frac{1}{\sigma}q(x_1,x_2,t), \; c^2 = S/_\sigma, \quad \sigma = \rho h, \tag{4.2}$$

$$-c^2\frac{\partial^4 w}{\partial x^4} - \ddot{w} = -\frac{1}{\sigma}q(x,t), \qquad c^2 = EI/_\sigma, \; \sigma = A\rho, \tag{4.3}$$

$$-c^2\nabla^4 w - \ddot{w} = -\frac{1}{\sigma}q(x_1,x_2,t), \; c^2 = N/_\sigma, \quad \sigma = \rho h. \tag{4.4}$$

They describe the transverse vibrations of strings, membranes, beams and plates, respectively. Only Equations (4.1) and (4.2) are hyperbolic. These equations are to be completed by the boundary and initial conditions.

Let us write Equation (4.1) - (4.4) in unified notation

$$\mathcal{D}(w) - \ddot{w} = -\frac{1}{\sigma}q(\underset{\sim}{x},t) \quad \underset{\sim}{x}\in A, \; t > 0. \tag{4.5}$$

This equation is completed by the boundary conditions appropriate for each of the system, and the following initial conditions:

$$w(\underset{\sim}{x},0) = f(\underset{\sim}{x}), \qquad \dot{w}(\underset{\sim}{x},0) = g(\underset{\sim}{x}), \; \underset{\sim}{x} \in A, \; t = 0. \tag{4.6}$$

Equation (4.5) and conditions (4.6) concern two-dimensional problems. However, in the case when the deflection is independent of the variable x_2 the above equation becomes one-dimensional.

We introduced before the Green's function $G(\underset{\sim}{x},\underset{\sim}{x}',t)$ satisfying the differential equation

$$\mathcal{D}(G) - \ddot{G} = -\frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t), \quad \underset{\sim}{x} \in A, \; t > 0, \tag{4.7}$$

with the same boundary conditions as for the function w of Equation (4.5) and with homogeneous initial conditions

$$G(\underset{\sim}{x},\underset{\sim}{x}',0) = 0, \quad \dot{G}(\underset{\sim}{x},\underset{\sim}{x}',0) = 0, \quad \underset{\sim}{x} \in A,\ t = 0. \tag{4.8}$$

The Green's function can be regarded as the deflection due to the action of an instantaneous concentrated external loading with intensity equal to unity.

Taking the Laplace transform of both sides of Equations (4.5) and (4.7) we obtain the equations

$$\mathcal{D}(\overline{w}) - (p^2\overline{w} - pf - g) = -\frac{1}{\sigma}\,\overline{q}(\underset{\sim}{x},p), \tag{4.9}$$

$$\mathcal{D}(\overline{G}) - p^2\overline{G} = -\frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{4.10}$$

We have made use here of the initial conditions (4.6) and (4.8) and we have introduced the notations

$$\overline{w}(\underset{\sim}{x},p) = \int_0^\infty w(\underset{\sim}{x},t)e^{-pt}dt, \quad \overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p) = \int_0^\infty G(x,x',t)e^{-pt}dt.$$

Let us multiply Equation (4.9) by $\overline{G}$ and Equation (4.10) by $\overline{w}$, subtract the results and integrate over the region A of the two-dimensional system. Then we obtain

$$\iint_A [\overline{G}\,\mathcal{D}(\overline{w}) - \overline{w}\,\mathcal{D}(\overline{G})]\,dA = -\frac{1}{\sigma}\iint_A \overline{q}(\underset{\sim}{x},p)\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p)dA(\underset{\sim}{x})$$

$$-\iint_A (pf(\underset{\sim}{x})+g(\underset{\sim}{x}))\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p)dA(\underset{\sim}{x}) + \frac{1}{\sigma}\iint_A \delta(\underset{\sim}{x}-\underset{\sim}{x}')\overline{w}(\underset{\sim}{x},p)dA(\underset{\sim}{x}).$$

Making use of the well-known theorem on the Dirac function

$$\iint_A \delta(\underset{\sim}{x}-\underset{\sim}{x}')\,f(\underset{\sim}{x})dA(\underset{\sim}{x}) = f(\underset{\sim}{x}'),$$

we arrive at the following formula for the transform of the deflection:

$$\bar{w}(\underset{\sim}{x}',p) = \iint_A \bar{q}(\underset{\sim}{x},p)\bar{G}(\underset{\sim}{x},\underset{\sim}{x}',p)\mathrm{d}A(\underset{\sim}{x})$$

$$+ \sigma\iint_A \left[g(\underset{\sim}{x})+pf(\underset{\sim}{x})\right]\bar{G}(\underset{\sim}{x},\underset{\sim}{x}',p)\mathrm{d}A(\underset{\sim}{x})+\sigma\iint_A \left[\bar{G}\,\mathcal{B}(\bar{w})-\bar{w}\,\mathcal{B}(\bar{G})\right]\mathrm{d}A(\underset{\sim}{x}). \tag{4.11}$$

We shall prove below that the last surface integral transformed into a curvilinear integral over the contour of the two-dimensional system, vanishes in view of the homogeneous boundary conditions. Thus, after inverting the Laplace transform and replacing $\underset{\sim}{x}$ by $\underset{\sim}{x}'$ we arrive at the integral expression

$$w(\underset{\sim}{x},t) = \int_0^t \mathrm{d}\tau\iint_A q(\underset{\sim}{x}',t-\tau)G(\underset{\sim}{x}',\underset{\sim}{x},\tau)\mathrm{d}A(\underset{\sim}{x}')$$

$$+ \sigma\iint_A \left[g(\underset{\sim}{x}')+f(\underset{\sim}{x}')\frac{\partial}{\partial t}\right]G(\underset{\sim}{x}',\underset{\sim}{x},t)\mathrm{d}A(\underset{\sim}{x}'). \tag{4.12}$$

For a one-dimensional problem we obtain from (4.12) the formula

$$w(x,t) = \int_0^t \mathrm{d}t\int_0^l q(x',t-\tau)G(x',x,\tau)\mathrm{d}x'$$

$$+ \sigma\int_0^l \left[g(x')+f(x')\,\frac{\partial}{\partial t}\right]G(x',x,t)\mathrm{d}x'. \tag{4.13}$$

In the above integral expressions the functions q, f, g are known. The knowledge of the function $G(x,x',t)$ makes it possible to determine by a simple integration over the variables x and t, the deflection of the considered system.

Let us now prove that the expression

$$I_1 = \int_0^l \left[\overline{G}(x,x',p)\,\mathcal{D}(\overline{w}(x,p)) - \overline{w}(x,p)\,\mathcal{D}(\overline{G}(x,x',p))\right]dx \tag{4.14}$$

in case of the one-dimensional problem, and the expression

$$I_2 = \iint_A \left[\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p)\,\mathcal{D}(\overline{w}(\underset{\sim}{x},p)) - \overline{w}(\underset{\sim}{x},p)\,\mathcal{D}(\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p))\right]dA(\underset{\sim}{x}), \tag{4.15}$$

vanish for the assumed homogeneous boundary conditions. Thus, in the case of transverse vibrations of a string $\mathcal{D}(w) = c^2\frac{\partial^2 w}{\partial x^2}$.

Integrating by parts we obtain

$$I_1 = c^2\int_0^l \left(\overline{G}\frac{d^2\overline{w}}{dx^2} - \overline{w}\frac{d^2\overline{G}}{dx^2}\right)dx = c^2\left[\overline{G}\frac{d\overline{w}}{dx} - \overline{w}\frac{d\overline{G}}{dx}\right]_0^l. \tag{4.16}$$

If the string is clamped at the cross-sections $x = 0,\ l$, then $\overline{w} = 0$, $\overline{G} = 0$. Hence $I_1 = 0$. For transverse vibrations of a beam $\mathcal{D}(w) = -c^2\frac{\partial^4 w}{\partial x^4}$. Consequently, the integral expression (4.14) yields after integration

$$I_1 = -c^2\int_0^l \left(\overline{G}\frac{d^4\overline{w}}{dx^4} - \overline{w}\frac{d^4\overline{G}}{dx^4}\right)dx = -c^2\left[\overline{G}\overline{w}''' - \overline{G}'\overline{w}'' + \overline{G}''\overline{w}' - \overline{G}'''\overline{w}\right]_0^l. \tag{4.17}$$

The expression in parenthesis vanishes for all types of the boundary conditions. If the beam is simply supported, then $\overline{w} = 0$, $\overline{w}'' = 0$, $\overline{G} = 0$, $\overline{G}'' = 0$; if it is clamped, then $\overline{w} = 0$, $\overline{w}' = 0$, $\overline{G} = 0$, $\overline{G}' = 0$. Finally for a free end of the beam we have $\overline{w}'' = 0$, $\overline{w}''' = 0$, $\overline{G}'' = 0$, $\overline{G}''' = 0$.

Let us now proceed to transverse vibrations of a membrane. Here $\mathcal{D}(w) = c^2\nabla^2 w$ and therefore

$$I_2 = c^2\iint_A (\overline{G}\nabla^2\overline{w} - \overline{w}\nabla^2\overline{G})\,dA = c^2\int_c \left(\overline{G}\frac{\partial\overline{w}}{\partial n} - \overline{w}\frac{\partial\overline{G}}{\partial n}\right)ds = 0 \tag{4.18}$$

in view of the Green transformation on the plane. The integral (4.18) vanishes in the case of a membrane supported on its boundary, for then $\overline{w} = 0$, $\overline{G} = 0$.

Consider finally the transverse vibrations of a plate. In this case $\mathfrak{D}(w) = -c^2\nabla^4 w$ and hence

$$I_2 = c^2 \iint_A (\overline{w}\nabla^4\overline{G} - \overline{G}\nabla^4\overline{w})\mathrm{d}A. \tag{4.19}$$

Transforming the above surface integral into a curvilinear integral over the boundary c of the plate we arrive at the expression

$$I_2 = -c^2 \int_c \left\{ M_{nn}(\overline{w})\frac{\partial\overline{G}}{\partial n} - V_n(\overline{w})\overline{G} - M_{nn}(\overline{G})\frac{\partial\overline{w}}{\partial n} + V_n(\overline{G})\overline{w} \right\} \mathrm{d}s \tag{4.20}$$

We have made use here of the transformations utilized in the derivation of the differential equation for the plate deflection and the formulae (3.51) and (3.54).

If the plate is simply supported on its boundary, then $\overline{w} = 0$, $M_{nn}(\overline{w}) = 0$ and $\overline{G} = 0$, $M_{nn}(\overline{G}) = 0$. If it is clamped, we have $\overline{w} = 0$, $\frac{\partial\overline{w}}{\partial n} = 0$ and $\overline{G} = 0$, $\frac{\partial\overline{G}}{\partial n} = 0$. Finally, if the boundary is free of tractions $M_{nn}(\overline{w}) = 0$, $V_n(\overline{w}) = 0$ and $M_{nn}(\overline{G}) = 0$, $V_n(\overline{G}) = 0$.

Thus, in all cases of homogeneous boundary conditions the integrals I_1, I_2 vanish. The deflection of the system is determined from Equation (4.12) or (4.13). Consequently, we have reduced the solution of differential equations for elastic systems to the determination of the Green function.

5. The Green function for the transverse vibration of a string of finite extent.

Consider the differential equation of the Green function $G(x,x',t)$

$$\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)G(x,x',t) = -\frac{1}{\sigma}\delta(x-x')\delta(t) \tag{5.1}$$

with the boundary conditions

$$G(0,x',t) = 0,\ G(l,x',t) = 0, \tag{5.2}$$

and homogeneous initial conditions

$$G(x,x',0) = 0,\ \dot{G}(x,x',0) = 0. \tag{5.3}$$

Applying the Laplace transform to the Equation (5.1), we obtain

$$\left(c^2\frac{d^2}{dx^2} - p^2\right)\overline{G}(x,x',p) = \frac{1}{\sigma}\delta(x-x'). \tag{5.4}$$

Now perform over (5.4) the finite sine transform:

$$\overline{G}(x,x',p) = \frac{2}{l}\sum_{n=1}^{\infty} G^*(n,x',p)\sin\alpha_n x \tag{5.5}$$

$$G^*(n,x',p) = \int_0^l \overline{G}(x,x',p)\sin\alpha_n x\,dx, \qquad \alpha_n = \frac{n\pi}{l}. \tag{5.6}$$

Multiplying both sides of Equation (5.4) by $\sin\alpha_n x$ and integrating from 0 to l, we obtain

$$\int_0^l \left(c^2\frac{d^2}{dx^2} - p^2\right)\overline{G}(x,x',p)\sin\alpha_n x\,dx = -\frac{1}{\sigma}\int_0^l \delta(x-x')\sin\alpha_n x\,dx \tag{5.7}$$

$$= -\frac{1}{\sigma}\sin\alpha_n x'.$$

Integrating by parts gives

$$\int_0^l \frac{d^2\overline{G}}{dx^2}\sin\alpha_n x\,dx = \alpha_n\left[(-1)^{n+1}\overline{G}(l,x',p)+\overline{G}(0,x',p)\right]-\alpha_n^2 G^*(n,x',p). \tag{5.8}$$

The quantity in square brackets on the right side of relations (5.8) vanishes since the boundary conditions are homogeneous.

Introducing the notations (5.5) we obtain

$$(c^2\alpha_n^2+p^2)G^*(n,x',p) = \frac{1}{\sigma}\sin\alpha_n x'. \tag{5.9}$$

Let us now invert the finite sine transform

$$\overline{G}(x,x',p) = \frac{2}{l\sigma}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x'\sin\alpha_n x}{p^2+c^2\alpha_n^2}, \tag{5.10}$$

and subsequently the Laplace transform. Taking into account that

$$\mathcal{L}^{-1}\left(\frac{1}{\alpha_n^2c^2+p^2}\right) = \frac{1}{\alpha_n c}\sin\alpha_n ct.$$

we are led to the solution in the series form

$$G(x,x',t) = \frac{2c}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x'\sin\alpha_n x}{\alpha_n}\sin\omega_n t,\ \omega_n = \alpha_n c. \tag{5.11}$$

Consider the differential equation of the deflection of the string

$$\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\omega(x,t) = -\frac{1}{\sigma}q(x,t) \tag{5.12}$$

with appropriate boundary conditions, and initial conditions

$$w(0,t) = w(l,t) = 0,\ w(x,0) = f(x),\ \dot{w}(x,0) = g(x). \tag{5.13}$$

The solution of the Equation (5.12) has the form

$$w(x,t) = \int_0^t d\tau\int_0^l q(x',\tau)G(x',x,t-\tau)dx' + \sigma\int_0^l\left[g(x')+f(x')\frac{\partial}{\partial t}\right]G(x',x,t)dx'. \tag{5.14}$$

Consider the case of the forced vibrations ($q\neq 0$, $f=0$, $g=0$):

$$w(x,t) = \int_0^t d\tau\int_0^l q(x',t-\tau)G(x',x,\tau)dx'. \tag{5.15}$$

Suppose that at ξ there acts a concentrated force which varies in time, viz.

$$q(x,t) = F(t)\delta(x-\xi), \quad 0 < x, \xi < l. \tag{5.16}$$

Introducing (5.16) into (5.15), we obtain

$$w(x,t) = \int_0^t F(\tau)G(\xi,x,t-\tau)d\tau. \tag{5.17}$$

If $F(t) = H(t)$, where $H(t)$ is the Heaviside function, i.e.

$$H(t) = \begin{cases} 0 \text{ for } t < 0, \\ 1 \text{ for } t > 0, \end{cases}$$

we obtain for the particular case

$$w(x,t) = \frac{2}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x}{\alpha_n}(1-\cos\omega_n t). \tag{5.18}$$

Locate now at ξ an external periodic concentrated force

$$q(x,t) = \delta(x-\xi)\cos\omega t, \quad \omega \neq \omega_n. \tag{5.19}$$

Introducing (5.19) into (5.17) we obtain the following formula

$$w(x,t) = \frac{2c^2}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x\sin\alpha_n\xi}{\omega_n^2-\omega^2}(\cos\omega t-\cos\omega_n t). \tag{5.20}$$

As $\omega \to \omega_n$, this relation takes the indeterminate form $\frac{0}{0}$. Applying the L'Hospital rule, we obtain

$$w(x,t) = \frac{c^2 t}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x\sin\alpha_n\xi}{\omega_n}\sin\omega_n t. \tag{5.21}$$

Formula (4.21) yields a steady increase of the deflection in time. It is valid only for small values of t, and hence for small deflections of the spring from equilibrium. This restriction is necessary, since the differential equation of deflection of string was derived under the assumption of small deflections as compared with the length of the string.

Consider one more particular case of loading of the string. Suppose that a force

$$q(x,t) = \begin{cases} H(t)\delta(x-Vt) & \text{for } 0 < Vt < l \\ 0 & \text{for } \quad Vt > l \end{cases} \tag{5.22}$$

is moving along the string with a constant velocity V, from $x = 0$ to $x = l$. Introducing (5.22) into formula (5.15), we finally obtain

$$w(x,t) = \frac{2c}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x}{\alpha_n(\alpha_n{}^2V^2-\omega_n{}^2)}(\alpha_n V\sin\omega_n t-\omega_n\sin\alpha_n Vt). \tag{5.23}$$

This formula is valid for $0 < Vt < l$. It yields the deflection at x due to the action of a concentrated force moving along the string with constant velocity V. It is readily observed that as $V \to 0$, $Vt \to \xi$ we pass from the dynamic to the static problem.

From (5.23) we obtain

$$w(x) = \frac{2}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi\sin\alpha_n x}{\alpha_n{}^2}. \tag{5.24}$$

In the following we assume that $q = 0$, $g = 0$ and $f \neq 0$. From the equation (5.14) and (5.11) we have

$$w(x,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin\alpha_n x\cos\omega_n t\int_0^l f(x')\sin\alpha_n x'\mathrm{d}x'. \tag{5.25}$$

Suppose that there acts on the string a concentrated force P at a point ξ. At time $t = 0$ we suddenly remove this static loading and the string begins free vibrations. The deflection of the string is given, for $t > 0$, by formula (5.25), in which we have to set

$$f(x) = w_0(x) = \frac{2P}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x\sin\alpha_n\xi}{\alpha_n{}^2} \tag{5.26}$$

Introducing (5.26) into (5.25) we obtain

$$w(x,t) = \frac{2P}{Sl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x}{\alpha_n^2}\sin\alpha_n\xi\cos\omega_n t,\ \omega_n = \alpha_n c. \tag{5.27}$$

6. The Green's function for the transverse vibrations of a membrane of finite extent.

Consider the Green's function $G(\underset{\sim}{x},\underset{\sim}{x}',t)$ satisfying the differential equation

$$(c^2\nabla^2-\partial^2 t)G(\underset{\sim}{x},\underset{\sim}{x}',t) = -\frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t),\ c^2 = \frac{S}{\sigma},\ \sigma=\rho h,\ \underset{\sim}{x}\equiv(x_1,x_2) \tag{6.1}$$

with homogeneous initial and boundary conditions.

First let us discuss the problem of vibrations of a rectangular plate.

Applying the Laplace transform to the Equation (6.1) we obtain

$$(c^2\nabla^2-p^2)\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p) = -\frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{6.2}$$

Now perform over (6.2) the finite sine transform. Introducing the notations

$$G^*(n,m;x_1',x_2';p) = \int_0^{a_1}\int_0^{a_2}\overline{G}(x_1,x_2;x_1',x_2';p)\sin\alpha_n x_1\sin\beta_m x_2 dx_1 dx_2, \tag{6.3}$$

$$\overline{G}(x_1,x_2;x_1',x_2';p) = \frac{4}{a_1a_2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}G^*(n,m;x_1',x_2';p)\sin\alpha_n x_1\sin\beta_m x_2, \tag{6.4}$$

$$\alpha_n = \frac{n\pi}{a_1},\qquad \beta_m = \frac{m\pi}{a_2},$$

we obtain the following form of Equation (6.2)

$$\left[c^2(\alpha_n{}^2+\beta_m{}^2)+p^2\right]G^*(n,m;x_1',x_2';p) = \frac{1}{\sigma}\sin\alpha_n x_1'\sin\alpha_m x_2'.$$

Inverting the finite sine transform in (6.5) gives

$$\overline{G}(x,x',p) = \frac{4}{a_1a_2\sigma}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\alpha_n x_1'\sin\beta_m x_2'}{c^2(\alpha_n{}^2+\beta_m{}^2)+p^2}\sin\alpha_n x_1\sin\beta_m x_2. \tag{6.6}$$

Inverting the Laplace transform in (6.6) gives finally

$$G(\underset{\sim}{x},\underset{\sim}{x}',t) = \frac{4c}{a_1a_2S}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\alpha_n x_1'\sin\beta_m x_2'}{\gamma_{nm}}\sin\alpha_n x_1\sin\beta_m x_2\sin(\gamma_{nm}ct) \tag{6.7}$$

$$\gamma_{nm} = (\alpha_n{}^2+\beta_m{}^2)^{\frac{1}{2}}.$$

Consider the case of forced vibrations. From the Equation (4.12) we have

$$w(\underset{\sim}{x},t) = \int_0^t d\tau\iint_A q(\underset{\sim}{x}',\tau)G(\underset{\sim}{x}',\underset{\sim}{x},t-\tau)dA(\underset{\sim}{x}'). \tag{6.8}$$

Suppose that a force

$$q(\underset{\sim}{x},t) = \begin{cases} F(t)\delta(x_1-Vt)\delta(x_2-\eta_2) & \text{for } 0 < Vt < l, \\ 0 & \text{for } \quad Vt > l \end{cases} \tag{6.9}$$

is moving along the membrane with a constant velocity V, from $x_1 = 0$ to $x_1 = a_1$ (along the line $x_2 = \eta_2$). Introducing (6.9) to Equation (6.8) we obtain

$$w(\underset{\sim}{x},t) = \int_0^t F(\tau)G(V\tau,\eta_2;x_1,x_2;t-\tau)d\tau. \tag{6.10}$$

If $F(t) = H(t)$, after integration with respect to t we have

$$w(\underset{\sim}{x},t) = \frac{4P_0c}{a_1a_2S}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\alpha_n x_1\sin\beta_m x_2\sin\beta_m\eta_2}{\gamma_{nm}(\alpha_n{}^2V^2-c^2\gamma_{nm}{}^2)}(\alpha_n V\sin\gamma_{nm}ct-c\gamma_{nm}\sin\alpha_n Vt) \tag{6.11}$$

Consider the problem of free and forced vibrations of a circular membrane. Assume that the forced and free vibrations depend only on the variable r. We have to do with the axisymmetric problem of vibrations. The solution of the differential equation

$$(c^2\nabla^2-\partial_t^2)w(r,t) = -\frac{1}{\sigma}q(r,t),\ \nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r},\ r = (x_1^2+x_2^2)^{\frac{1}{2}}, \tag{6.12}$$

has the form

$$w(r,t) = \int_0^a dr'\int_0^t q(r',\tau)G(r',r,t-\tau)d\tau+\sigma\int_0^a\left[g(r')+f(r')\frac{\partial}{\partial t}\right]G(r',r,t)dr'. \tag{6.13}$$

If the Green's function $G(r,r',t)$ is determined, the deflection $w(r,t)$ can be found from Equation (6.13).

The differential equation of the Green's function can be written in the cylindrical coordinates

$$\left[c^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)-\frac{\partial^2}{\partial t^2}\right]G(r,r',t) = -\frac{1}{\sigma}\delta(r-r')\delta(t). \tag{6.14}$$

We assume that the boundary condition and the initial conditions are homogeneous. Let us perform over the differential equation (6.14) the Laplace transform, whence

$$\left[c^2\left(\frac{d^2}{dr^2}+\frac{}{r}\frac{d}{dr}\right)-p^2\right]\overline{G}(r,r',p) = -\frac{1}{\sigma}\delta(r-r'). \tag{6.15}$$

We can solve the Equation (6.15) with use of the finite Hankel transform

$$G^*(n,r',p) = \int_0^a \overline{G}(r,r',p)rJ_0(\alpha_n r)dr, \tag{6.16}$$

$$\overline{G}(r,r',p) = \frac{2}{a^2}\sum_{n=1}^{\infty}G^*(n,r',p)\frac{J_0(\alpha_n r)}{[J_1(\alpha_n a)]^2}. \tag{6.17}$$

The parameter α_n should satisfy the transcendental equation

$$J_0(\alpha_n a) = 0, \quad n = 1,2,\ldots,\infty. \tag{6.18}$$

Multiply Equation (6.15) throughout by $rJ_0(\alpha_n r)$ and integrate with respect to r from 0 to a.

$$\int_0^a \left[c^2\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) - p^2\right]\overline{G}rJ_0(\alpha_n r)dr = -\frac{1}{\sigma}\int_0^a \delta(r-r')rJ_0(\alpha_n r)dr. \tag{6.19}$$

Perform the integration by parts

$$\int_0^a r\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\overline{G}J_0(\alpha_n r)dr = \left[r\frac{d\overline{G}}{dr}J_0(\alpha_n r) - \alpha_n r\overline{G}J_0(\alpha_n r)\right]_0^a$$

$$- \alpha_n^2\int_0^a r\overline{G}J_0(\alpha_n r)dr. \tag{6.20}$$

The expression in brackets vanishes for the upper limit, provided $J_0(\alpha_n a) = 0$; for $r = 0$ it vanishes always.

Taking into account Equation (6.20) we find that

$$c^2(\alpha_n^2 + p^2)G^*(n,r',p) = \frac{1}{\sigma}r'J_0(\alpha_n r'). \tag{6.21}$$

Performing the inverse Hankel transformation on (6.17), we obtain

$$\overline{G}(r,r',p) = \frac{2}{a^2\sigma}\sum_{n=1}^{\infty}\frac{r'J_0(\alpha_n r')J_0(\alpha_n r)}{(\alpha_n^2c^2+p^2)\left[J_1(\alpha_n a)\right]^2}. \tag{6.22}$$

Applying the inverse Laplace transformation, we arrive finally at the result

$$G(r,r',t) = \frac{2}{a^2\sigma}\sum_{n=1}^{\infty}\frac{r'J_0(\alpha_n r')J_0(\alpha_n r)}{\alpha_n c\left[J_1(\alpha_n a)\right]^2}\sin\omega_n t, \quad \omega_n = \alpha_n c. \tag{6.23}$$

In the particular case of a concentrated load $q(r,t) = \frac{P_0}{2\pi r}\delta(r)H(t)$

we obtain

$$w(r,t) = \frac{P_0}{\pi a^2 S}\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r)}{\alpha_n^2\left[J_1(\alpha_n a)\right]^2}(1-\cos\omega_n t). \tag{6.24}$$

7. Free vibrations of an infinite string and an infinite membrane.

Consider the homogeneous differential equation of the transverse vibrations of the string (4.1). Assume that the string is infinitely long, and that its motion is determined by the initial conditions

$$w(x,0) = f(x), \quad \dot{w}(x,0) = g(x), \tag{7.1}$$

which mean that at the time $t = 0$ the string has deflection $f(x)$ and velocity $g(x)$.

The solution of the equation

$$c^2\frac{\partial^2 w}{\partial x^2} - \ddot{w} = 0 \tag{7.2}$$

has the form

$$w(x,t) = \sigma\int_{-\infty}^{\infty}\left[g(x')+f(x')\frac{\partial}{\partial t}\right]G(x',x,t)\mathrm{d}x'. \tag{7.3}$$

Thus, we have to solve the equation

$$\left(c^2\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)G(x,x',t) = -\frac{1}{\sigma}\delta(x-x')\delta(t), \quad -\infty<x<\infty, \quad t>0, \tag{7.4}$$

with the homogeneous initial conditions and boundary conditions in infinity: $G \to 0$ for $|x| \to \infty$.

Performing the Laplace transform over Equation (7.4) for the above homogeneous initial conditions, we obtain

$$\left(c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} - p^2\right)\overline{G}(x,x',p) = -\frac{1}{\sigma}\delta(x-x'). \tag{7.5}$$

Further, perform over Equation (7.5) the exponential Fourier transform. Multiplying both sides of Equation (7.5) by $\frac{1}{\sqrt{2\pi}}e^{i\xi x}$ and integrating from $-\infty$ to $+\infty$, we obtain

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(c^2\frac{d^2}{dx^2} - p^2)\overline{G}(x,x',p)e^{i\xi x}dx = -\frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x-x')e^{i\xi x}dx. \tag{7.6}$$

Integrating the first term of (7.6) by parts gives

$$\int_{-\infty}^{\infty}\frac{d\overline{G}}{dx}e^{i\xi x}dx = \left|e^{i\xi x}(\frac{d\overline{G}}{dx} - i\xi\overline{G})\right|_{-\infty}^{\infty} - \xi^2\int_{-\infty}^{\infty}\overline{G}e^{i\xi x}dx. \tag{7.7}$$

The quantity in square brackets on the right side of the relation (7.7) vanishes, since at infinity both the deflection $\overline{G}$ and its derivative $\frac{d\overline{G}}{dx}$ vanish. Introducing the notations

$$\tilde{G}(\xi,x',p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{G}(x,x',p)e^{i\xi x}dx, \tag{7.8}$$

$$\overline{G}(x,x',p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{G}(\xi,x',p)e^{-i\xi x}d\xi, \tag{7.9}$$

we transform Equation (7.7) to the form

$$(c^2\xi^2+p^2)\tilde{G} = \frac{1}{\sigma\sqrt{2\pi}}e^{i\xi x'} \tag{7.10}$$

Introducing the Fourier transform for the expression $p\overline{G}$, we obtain

$$p\overline{G}(x,x',p) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\frac{p}{c^2\xi^2+p^2}e^{-i\xi(x-x')}d\xi. \tag{7.11}$$

Let us now invert the Laplace transform. Taking into account that

$$\mathcal{L}^{-1}\left(\frac{p}{p^2+c^2\xi^2}\right) = \cos\xi ct,$$

we obtain

$$\frac{\partial G}{\partial t} = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty} \cos\xi ct \; e^{-i\xi(x-x')}d\xi. \tag{7.12}$$

Moreover, in view of the relations

$$\cos\xi ct = \tfrac{1}{2}(e^{i\xi ct}+e^{-i\xi ct}), \quad \int_{-\infty}^{\infty} e^{-i\eta\xi}d\xi = 2\pi\delta(\eta)$$

we obtain the final for the solution (7.12)

$$\frac{\partial G}{\partial t} = \frac{1}{2\sigma}\left[\delta(x-x'-ct)+\delta(x-x'+ct)\right]. \tag{7.13}$$

Assume now, that $\dot{w}(x,0) = g(x) = 0$. Introducing (7.13) into the integral expression (7.3) we have [93]

$$w(x,t) = \tfrac{1}{2}\int_{-\infty}^{\infty} f(x')\left[\delta(x-x'-ct)+\delta(x-x'+ct)\right]dx'$$

$$= \tfrac{1}{2}\left[f(x-ct)+f(x+ct)\right]. \tag{7.14}$$

This is the d'Alembert solution of the wave equation of the string. The Equation (7.14) can be interpreted in the following way. Let us deflect the string to the form of the curve $f(x)$ at instant $t = 0$, and remove the forces which produced the initial deflection, without inducing an initial velocity of the element of the string (i.e. $g = 0$). For $t > 0$ the deflected form of the string is divided into two waves (the waves $\frac{1}{2}f(x-ct)$ and $\frac{1}{2}f(x+ct)$). One wave moves to the right with the constant velocity $c = (\frac{S}{\sigma})^{\frac{1}{2}}$, while the second moves to the left (Fig. 7.1).

Let us discuss the problem of free vibrations of an infinite membrane. The equation of the transverse vibration of the membrane has the form

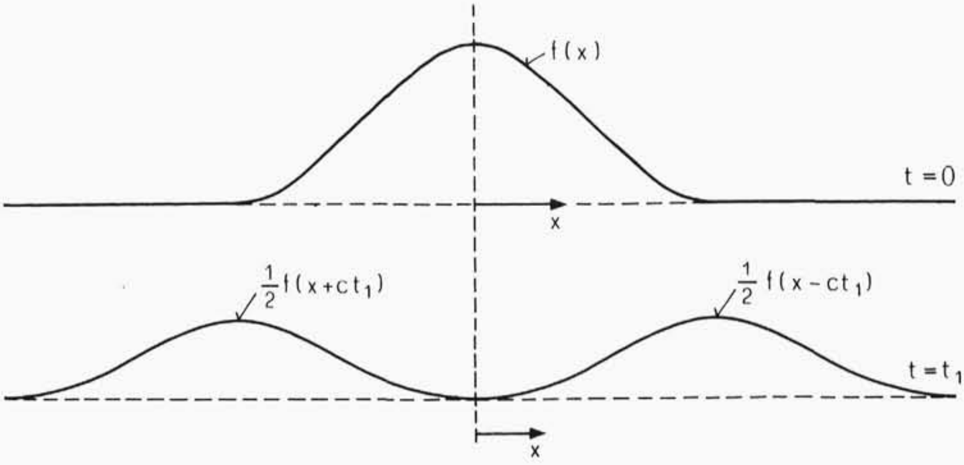

Fig. 7.1

$$c\nabla^2 w - \ddot{w} = 0, \; \underset{\sim}{x} \in A, \; t > 0. \tag{7.15}$$

Furthermore, we assume the following form for the initial conditions:-

$$w(\underset{\sim}{x},0) = f(\underset{\sim}{x}), \; \dot{w}(\underset{\sim}{x},0) = g(\underset{\sim}{x}), \; \underset{\sim}{x} \equiv (x_1,x_2) \in A, \; t = 0. \tag{7.16}$$

The solution of the differential Equation (7.15) takes the form

$$w(\underset{\sim}{x},t) = \sigma\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[g(\underset{\sim}{x}')+f(\underset{\sim}{x}')\frac{\partial}{\partial t}\Big]G(\underset{\sim}{x},\underset{\sim}{x}',t)\,dA(\underset{\sim}{x}'). \tag{7.17}$$

The differential equation

$$\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)G(\underset{\sim}{x},\underset{\sim}{x}',t) = -\frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t). \tag{7.18}$$

is to be solved with the homogeneous initial conditions. We first of all place the Dirac function at the origin of the coordinate system.

In this case the equation of transverse vibrations of the membrane can be written in cylindrical coordinates

$$(c^2\nabla_r^2-\partial_t^2)G(r,0,t) = -\frac{1}{\sigma}\frac{\delta(r)}{2\pi r}\delta(t), \tag{7.19}$$

where

$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r},\ c^2 = S/_\sigma.$$

Applying the Laplace transform to Equation (7.19) we have

$$(c^2\nabla_r^2-p^2)\overline{G}(r,0,p) = -\frac{1}{\sigma}\frac{\delta(r)}{2\pi r} \tag{7.20}$$

Denote by $\tilde{G}(\alpha,0,p)$ the Hankel transform of the function $\overline{G}(r,0,p)$:

$$\tilde{G}(\alpha,0,p) = \int_0^\infty \overline{G}(r,0,p)rJ_0(\alpha r)\mathrm{d}r, \tag{7.21}$$

and by $\overline{G}(r,0,p)$ the inverse Hankel transform

$$\overline{G}(r,0,p) = \int_0^\infty \tilde{G}(\alpha,0,p)\alpha J_0(\alpha r)\mathrm{d}\alpha. \tag{7.22}$$

Here we observe that

$$\int_0^\infty r\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\overline{G}(r,0,p)J_0(\alpha r)\mathrm{d}r = -\alpha^2\tilde{G}(\alpha,0,p). \tag{7.23}$$

Multiplying Equation (7.20) by $rJ_0(\alpha r)$, and integrating with respect to r over the interval $(0,\infty)$, we transform Equation (7.20) to the form

$$(c^2\alpha^2+p^2)\tilde{G}(\alpha,0,p) = \frac{1}{2\pi\sigma}. \tag{7.24}$$

Applying the inverse Hankel transform (7.22), we have

$$\overline{G}(r,0,p) = \frac{1}{2\pi\sigma}\int_0^\infty \frac{\alpha J_0(\alpha r)\mathrm{d}\alpha}{p^2+\alpha^2c^2}. \tag{7.25}$$

Applying the inverse Laplace transform we arrive finally at the relation

$$G(r,0,t) = \frac{1}{2\pi\sigma c}\int_0^\infty J_0(\alpha r)\sin\alpha ct\,d\alpha \tag{7.26}$$

or

$$G(r,0,t) = \frac{1}{2\pi\sigma c}\begin{cases}(c^2t^2-r^2)^{-\frac{1}{2}} & \text{for } 0 < r < ct\\ 0 & \text{for } ct < r < \infty\end{cases} \tag{7.27}$$

where $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$.

Now, we remove the concentrated and instantaneous impulse to the point $\underset{\sim}{x}'$.

We obtain

$$G(\underset{\sim}{x},\underset{\sim}{x}',t) = \frac{1}{2\pi\sigma c}\begin{cases}(c^2t^2-r^2)^{-\frac{1}{2}} & \text{for } 0 < r < ct\\ 0 & \text{for } ct < r < \infty\end{cases} \tag{7.28}$$

where, now,

$$r = \left[(x_1-x_1') + (x_2-x_2')^2\right]^{\frac{1}{2}}$$

is the distance between the points $\underset{\sim}{x}$ and $\underset{\sim}{x}'$.

8. The Green's function $G(x,x',t)$ for the transverse vibrations of a rod.

Consider a rod of finite length l which undergoes forced and free vibrations. Assume that at time $t = 0$ the deflection and velocity of the rod are known, i.e. that

$$w(x,0) = f(x), \quad \dot{w}(x,0) = g(x). \tag{8.1}$$

Thus we have to solve the differential equation

$$c^2\frac{\partial^4 w}{\partial x^4} + \ddot{w} = \frac{1}{\sigma}q(x,t), \quad c^2 = \frac{EI}{\sigma}, \ \sigma = \rho A, \tag{8.2}$$

with the initial conditions (8.1) and homogeneous boundary conditions at the ends of the rod. The solution of the Equation (8.2) has the form

$$w(x,t) = \int_0^t d\tau \int_0^l q(x',t-\tau)G(x',x,\tau)dx' + \sigma\int_0^l \left[g(x')+f(x')\frac{\partial}{\partial t}\right]G(x',x,t)dx'. \quad (8.3)$$

The starting point of our considerations is to solve the differential equation of the Green function

$$\left(c\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2}\right)G(x,x',t) = \frac{1}{\sigma}\delta(x-x')\delta(t). \quad (8.4)$$

The Green function must satisfy the Equation (8.4) with the homogeneous initial conditions

$$G(x,x',0) = 0, \quad \dot{G}(x,x',0) = 0, \quad (8.5)$$

and the same boundary conditions as the function $w(x,t)$.

Now consider the differential equation

$$\frac{d^4W}{dx^4} = \lambda^4 W(x); \qquad \lambda^4 = \frac{\omega^2}{c^2}. \quad (8.6)$$

the differential equation of harmonic free transverse vibrations. Apply now to (8.6) the Laplace transform. The transformed equation (8.6) takes the form

$$(p^4-\lambda^4)\overline{W}(p) = p^3W(0) + p^2W'(0) + pW''(0) + W'''(0), \quad (8.7)$$

$$\overline{W}(p) = \int_0^\infty W(x)e^{-px}dx.$$

Now inverting the Laplace transform in Equation (8.7) and taking into account the relations

$$\frac{1}{p^4-\lambda^4} = \frac{1}{2\lambda^2}\left(\frac{1}{p^2-\lambda^2} - \frac{1}{p^2+\lambda^2}\right) \tag{8.8}$$

$$\mathcal{L}^{-1}\left(\frac{1}{p^4-\lambda^4}\right) = \frac{1}{\lambda^3}\left(\frac{\sinh\lambda x-\sin\lambda x}{2}\right) = \frac{1}{\lambda^3}V(\lambda x),$$

$$\mathcal{L}^{-1}\left(\frac{p}{p^4-\lambda^4}\right) = \frac{1}{\lambda^2}\left(\frac{\cosh\lambda x-\cos\lambda x}{2}\right) = \frac{1}{\lambda^2}U(\lambda x),$$

$$\mathcal{L}^{-1}\left(\frac{p^2}{p^4-\lambda^4}\right) = \frac{1}{\lambda}\left(\frac{\sinh\lambda x+\sin\lambda x}{2}\right) = \frac{1}{\lambda}T(\lambda x),$$

$$\mathcal{L}^{-1}\left(\frac{p^3}{p^4-\lambda^4}\right) = \frac{1}{2}\left(\cosh\lambda x+\cos\lambda x\right) = S(\lambda x),$$

We arrive at the following form of the solution of Equation (8.7):

$$W(x) = W(0)S(\lambda x)+\frac{1}{\lambda}W'(0)T(\lambda x)+\frac{1}{\lambda^2}W''(0)U(\lambda x)+\frac{1}{\lambda^3}W'''(0)V(\lambda x). \tag{8.9}$$

The solution (8.7) has a number of advantages. The constants $W(0)$, $W'(0)$, $W''(0)$, $W'''(0)$ appearing in it can be interpreted as the deflection, the angle of inclination of the tangent of the deformed rod, a quantity proportional to the bending moment and a quantity proportional to the shear force, all in the cross-section $x_1 = 0$. For arbitrary boundary conditions, two of these quantities vanish.

Differentiating the function $W(x)$ in formula (8.9), we obtain

$$W'(x) = W(0)\lambda V(\lambda x)+W'(0)S(\lambda x)+\frac{1}{\lambda}W''(0)T(\lambda x)+\frac{1}{\lambda^2}W'''(0)U(\lambda x),$$

$$W''(x) = W(0)\lambda^2U(\lambda x)+W'(0)\lambda V(\lambda x)+W''(0)S(\lambda x)+\frac{1}{\lambda}W'''(0)T(\lambda x), \tag{8.10}$$

$$W'''(x) = W(0)\lambda^3T(\lambda x)+W'(0)\lambda^2U(\ x)+W''(0)\lambda V(\lambda x)+W'''(0)S(\lambda x).$$

Making use of the solution (8.9) and the relations (8.10), we can, in a very simple way, determine the frequencies of vibrations and their corresponding modes.

Suppose that the rod is clamped at the cross-section $x = 0$ and simply supported at $x = l$. The boundary conditions therefore are the following:

$$W(0) = 0, \quad W'(0) = 0, \quad W(l) = 0, \quad W''(l) = 0. \tag{8.11}$$

In view of (8.9) and the second relation (8.10), and taking into account the boundary conditions (8.11), we are led to the system of two equations

$$\frac{1}{\lambda^2}W''(0)U(\lambda l) + \frac{1}{\lambda^3}W'''(0)V(\lambda l) = 0,$$
$$W''(0)S(\lambda l) + \frac{1}{\lambda}W'''(0)T(\lambda l) = 0.$$

Equating to zero the determinant of this system, we have

$$\tanh\beta - \tan\beta = 0, \quad \beta = \lambda l. \tag{8.12}$$

This is a transcendental equation having an infinite number of roots. The first five are the following:

$$\beta_1 = 3{,}927,\ \beta_2 = 7{,}069,\ \beta_3 = 10{,}210,\ \beta_4 = 13{,}352,\ \beta_5 = 16{,}483$$
$$\beta_r = \frac{\pi}{4}(4r+1), \quad r > 5.$$

Since

$$\frac{\omega^2}{c^2} = \lambda^4, \qquad \beta = \lambda l, \qquad c^2 = \frac{EI}{\sigma}, \qquad \sigma = \rho A.$$

the consecutive frequencies of vibration have the form

$$\omega_n = \frac{\beta_n{}^2 c}{l^2} = \frac{\beta_n{}^2}{l^2}\sqrt{\frac{EI}{\sigma}}, \qquad n = 1,2,\ldots,\infty.$$

The mode of free vibration $W_n(x_1)$, corresponding to the frequency ω_n is given by the formula (8.9).

Since $W(0) = W'(0) = 0$, we have

$$W_n(x) = \frac{1}{\lambda_n^2} W''(0)U(\lambda_n x) + \frac{1}{\lambda_n^3} W'''(0)V(\lambda_n x). \tag{8.14}$$

We now prove that the modes of free vibration possess the important property of orthogonality. Denote by ω_k, ω_l the frequencies and by $W_k(x)$, $W_l(x)$ the corresponding modes of vibration. Suppose that both vibrations satisfy the same boundary conditions. The functions $W_k(x)$, $W_l(x)$ satisfy the equations

$$\frac{d^4W_k}{dx^4} - \lambda_k^4 W_k = 0, \quad \frac{d^4W_l}{dx^4} - \lambda_l^4 W_l = 0. \tag{8.15}$$

From these equations we obtain

$$\int_0^l (W_l\frac{d^4W_k}{dx^4} - W_k\frac{d^4W_l}{dx^4})dx = (\lambda_k^4 - \lambda_l^4)\int_0^l W_kW_l dx$$

or

$$\left[W_lW_k''' - W_l'W_k'' + W_l''W_k' - W_l'''W_k\right]_0^l = (\lambda_k^4 - \lambda_l^4)\int_0^l W_kW_l dx.$$

The expression in square brackets vanishes for both limits:

$$(\lambda_k^4 - \lambda_l^4)\int_0^l W_kW_l dx = 0. \tag{8.16}$$

Since $\lambda k \neq \lambda l (\omega k \neq \omega l)$, which has been assumed in view of two different forms of vibration, Equation (8.16) is satisfied only if

$$\int_0^l W_k(x)W_l(x)dx = 0 \quad k \neq l. \tag{8.17}$$

This is the condition of orthogonality of the modes of free vibration of a rod. Consider the integral

$$\int_0^l [W_k(x)]^2 dx = \gamma. \tag{8.18}$$

Bearing in mind that the mode of vibration contains a constant C, we choose the latter in such a way that integral (8.18) equals unity.

In the case of the vibration of a rod simply supported at both ends, we have

$$C^2\int_0^l \sin^2\alpha_k x\,dx = \gamma, \quad \alpha_k = \frac{k\pi}{l},$$

whence $\frac{C^2 l}{2} = \gamma$. If, therefore, we set $C = \sqrt{\frac{2}{l}}$, we obtain $\gamma = 1$.

The functions

$$W_k(x) = \sqrt{\frac{2}{l}}\sin\alpha_k x, \quad \alpha_k = \frac{k\pi}{l},$$

are called the normalised functions of free vibrations of a rod simply supported at both ends.

In subsequent considerations we assume that the eigenfunctions $W_k(x)$ satisfy the condition

$$\int_0^l W_k(x)W_l(x)\,dx = \delta_{kl} = \begin{cases} 0 \text{ if } k \neq l \\ 1 \text{ if } k = l. \end{cases} \quad (8.19)$$

Consider now the differential equation (8.4).

Apply the Laplace transform to Equation (8.4), whence

$$\left(c^2\frac{d^4}{dx^4} + p^2\right)\overline{G}(x,x',p) = \frac{1}{\sigma}\delta(x-x'). \quad (8.20)$$

Let us now expand the function $\overline{G}$ into an infinite series of the eigenfunctions $W_n(x)$, which satisfy the equation

$$\frac{d^4W_n}{dx^4} - \lambda_n^4 W_n = 0 \quad (8.21)$$

with the same boundary conditions as the function $\overline{G}$ and w. We introduce the notation

$$G^*(n,x',p) = \int_0^l \overline{G}(x,x',p)W_n(x)\mathrm{d}x, \tag{8.22}$$

$$\overline{G}(x,x',p) = \sum_{n=1}^{\infty} G^*(n,x',p)W_n(x). \tag{8.23}$$

This is a new finite transform. We have

$$\int_0^l \overline{G}(x,x',p)W_m(x)\mathrm{d}x = \sum_{n=1}^{\infty} G^*(n,x',p)\int_0^l W_m(x)W_n(x)\mathrm{d}x$$

$$= \sum_{n=1}^{\infty} G^*(n,x',p)\delta_{nm} = G^*(m,x',p).$$

We have used the condition of orthogonality for the functions $W_n(x)$. Now we multiply Equation (8.20) by $W_n(x)$ and integrate from 0 to l.

$$\int_0^l \left(c^2\frac{\mathrm{d}^4}{\mathrm{d}x^4} + p^2\right)\overline{G}W_n\mathrm{d}x = \frac{1}{\sigma}\int_0^l \delta(x-x')W_n(x)\mathrm{d}x. \tag{8.24}$$

Since

$$\int_0^l \frac{\mathrm{d}^4\overline{G}}{\mathrm{d}x^4}W_n\mathrm{d}x = \int_0^l \overline{G}\frac{\mathrm{d}^4W_n}{\mathrm{d}x^4}\mathrm{d}x + \left[W_n\overline{G}''' - W_n'\overline{G}'' + W_n''\overline{G}' - W_n'''\overline{G}\right]_0^l$$

$$= \lambda_n^4\int_0^l \overline{G}W_n\mathrm{d}x = \lambda_n^4 G^*(n,x',p), \tag{8.25}$$

we have

$$(c^2\lambda_n^4 + p^2)G^*(n,x',p) = \frac{1}{\sigma}W_n(x'). \tag{8.26}$$

We now apply the inverse transform (8.23) to obtain Equation (8.26)

$$\overline{G}(x,x',p) = \frac{1}{\sigma}\sum_{n=1}^{\infty}\frac{W_n(x')W_n(x)}{p^2+c^2\lambda_n^4} \tag{8.27}$$

Inverting the Laplace transform, we obtain

$$G(x,x',t) = \frac{1}{\sigma}\sum_{n=1}^{\infty}\frac{W_n(x')W_n(x)}{\omega_n}\sin\omega_n t,\ \omega_n = c\lambda_n^2. \tag{8.28}$$

Let us examine forced vibrations: $q \neq 0$, $f = 0$, $g = 0$. The Equation (8.3) takes the form

$$w(x,t) = \int_0^l dx'\int_0^t q(x',t-\tau)G(x',x,\tau)d\tau \tag{8.29}$$

$$= \int_0^l dx'\int_0^t q(x',\tau)G(x',x,t-\tau)d\tau.$$

Suppose that along the rod there moves a concentrated force of intensity $F(t)$ with the constant velocity V, i.e.

$$q(x,t) = \begin{cases} F(t)\delta(x-Vt) & \text{for } 0 < Vt < l, \\ 0 & \text{for } \quad Vt > l. \end{cases} \tag{8.30}$$

We have assumed that $q(x,t)$ varies in time during the motion along the rod. Introducing (8.30) into Equation (8.29) and taking into account that

$$\int_0^l \delta(x'-Vt)W_n(x')dx' = W_n(Vt),$$

we obtain for the delfection of the rod the formula

$$w(x,t) = \frac{1}{\sigma}\sum_{n=1}^{\infty}W_n(x)\int_0^t F(\tau)W_n(\tau V)\frac{\sin\omega_n(t-\tau)d\tau.}{\omega_n} \tag{8.31}$$

If $F(t) = P_0H(t)$, we have

$$w(x,t) = \frac{P_0}{\sigma}\sum_{n=1}^{\infty}\frac{W_n(x)}{\omega_n}\int_0^t W_n(V\tau)\sin\omega_n(t-\tau)\,d\tau \tag{8.32}$$

In the particular case of a rod simply supported at both ends, we obtain from (8.32)

$$w(x,t) = \frac{2P_0}{l\sigma}\sum_{n=1}^{\infty}\frac{\sin\alpha_n x}{\omega_n(\alpha_n^2V^2-\omega_n^2)}(\alpha_n V\sin\omega_n t-\omega_n\sin\alpha_n Vt). \tag{8.33}$$

This formula is valid for $0 < Vt < l$. Formula (8.33) is valid also for the case of a static concentrated force. It suffices to take $V \to 0$, and to assume that, in spite of the infinitely small velocity, the force reaches point ξ i.e. we set $V \to 0$, $Vt \to \xi$, $\sin\alpha_n Vt \to \sin\alpha_n\xi$ in (8.33). Hence

$$w_{st}(x) = \frac{2P_0}{EIl}\sum_{n=1}^{\infty}\frac{\sin\alpha_n\xi}{\alpha_n^4}\sin\alpha_n x. \tag{8.34}$$

Knowing the deflection of the rod $w(x,t)$, we can calculate the bending moment $M(x,t)$ and the shear force $T(x,t)$ by the formula

$$M = -EI\frac{\partial^2 w}{\partial x^2}, \quad T(x,t) = -EI\frac{\partial^3 w}{\partial x^3}. \tag{8.35}$$

9. The Green's function $G(\underset{\sim}{x},\underset{\sim}{x}',t)$ for the transverse vibration of a thin plate.

Let us investigate the problem: what frequencies ω and what vibrational modes lead to harmonic free vibration of a rectangular plate supported along the edges. Assume that

$$w(x_1,x_2,t) = W(x_1,x_2)e^{i\omega t} \tag{9.1}$$

and insert the function (9.1) into homogeneous equation governing the bending of the plate. This equation then takes the form

$$\nabla^4 W - \lambda^4 W = 0, \quad \lambda^4 = \frac{\omega^2}{c^2}, \quad c^2 = \frac{N}{\rho h}. \tag{9.2}$$

Assume, also, that $W(x_1, x_2) = X(x_1)Y(x_2)$, which corresponds with certain types of boundary condition of a rectangular plate. In such a case Equation (9.2) takes the form

$$X^{iv}(x_1)Y(x_2) + 2X''(x_1)Y''(x_2) + X(x_1)Y^{iv}(x_2) - \lambda^4 X(x_1)Y(x_2) = 0. \tag{9.3}$$

The functions $X(x_1)$, $Y(x_2)$ can be separated in the above equation for instance, provided that either

$$X''(x_1) = -\alpha^2 X(x_1), \quad X^{iv}(x_1) = -\alpha^2 X''(x_1), \tag{9.4}$$

or

$$Y''(x_2) = -\alpha^2 Y(x_1), \quad Y^{iv}(x_2) = -\alpha^2 Y''(x_2). \tag{9.5}$$

The conditions (9.4) and (9.5) are fulfilled only by trigonometric functions

$$\left\{\begin{matrix} \sin\alpha_n x_1 \\ \\ \cos\alpha_n x_1 \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} \sin\beta_m x_2 \\ \\ \cos\beta_m x_2 \end{matrix}\right\} \quad \text{where } \alpha_n = \frac{n\pi}{a_1}, \ \beta_m = \frac{m\pi}{a_2}.$$

We assume that the plate is simply supported on the edges $x_2 = 0$, a_2. This implies

$$Y_m(x_2) = C\sin\beta_m x_2, \qquad m = 1,2,\ldots,\infty, \tag{9.6}$$

since this function satisfies the conditions

$$Y_m(0) = Y_m(a_2) = 0, \ Y''_m(0) = Y''_m(a_2) = 0 \tag{9.7}$$

for any integer m, and hence also the boundary conditions

$$w(x_1,0,t) = w(x_1,a_2,t) = \nabla^2 w(x_1,0,t) = \nabla^2 w(x_1,a_2,t) = 0.$$

In the case under consideration, Equation (9.3) takes the form

$$\frac{d^4X}{dx_1^4} - 2\beta_m^2\frac{d^2X}{dx_1^2} - (\lambda^4-\beta_m^4)X = 0. \tag{9.8}$$

Apply the Laplace transform to Equation (9.8). We have

$$\overline{X}(p) = \frac{(p^2-2\beta_m^2)\left[pX(0)+X'(0)\right]+pX''(0)+X'''(0)}{(p^2-\beta_m^2)^2 - \lambda^4} \tag{9.9'}$$

i.e.

$$X(p) = \frac{1}{2\lambda^2}\left(\frac{1}{p^2-\delta^2} - \frac{1}{p^2+\varepsilon^2}\right)\left[pX''(0) + X'''(0)\right] \tag{9.9''}$$

$$+ \frac{1}{2\lambda^2}\left(\frac{\varepsilon^2}{p^2-\delta^2} + \frac{\delta^2}{p^2+\varepsilon^2}\right)(pX(0) + X'(0))$$

where $\varepsilon^2 = \lambda^2 - \beta_m^2$, $\delta^2 = \lambda^2 + \beta_m^2$.

Observing that

$$\mathcal{L}^{-1}\left(\frac{p}{p^2-\delta^2}\right) = \cosh\delta x_1, \quad \mathcal{L}^{-1}\left(\frac{p}{p^2+\varepsilon^2}\right) = \cos\varepsilon x_1,$$

$$\mathcal{L}^{-1}\left(\frac{1}{p^2-\delta^2}\right) = \frac{1}{\delta}\sinh\delta x_1, \quad \mathcal{L}^{-1}\left(\frac{1}{p^2+\varepsilon^2}\right) = \frac{1}{\varepsilon}\sin\varepsilon x_1,$$

we find that Equation (9.9"), by means of the inverse Laplace transform, yields

$$X(x_1) = X(0)A(x_1)+X'(0)B(x_1)+X''(0)C(x_1)+X'''(0)D(x_1), \tag{9.10}$$

where the following notations have been introduced:-

$$\left.\begin{aligned} A(x_1) &= \frac{1}{2\lambda^2}(\varepsilon^2\cosh\delta x_1+\delta^2\cos\varepsilon x_1), \\ B(x_1) &= \frac{1}{2\lambda^2}\left(\frac{\varepsilon^2}{\delta}\sinh\delta x_1 + \frac{\delta^2}{\varepsilon}\sin\varepsilon x_1\right), \\ C(x_1) &= \frac{1}{2\lambda^2}(\cosh\delta x_1 - \cos\varepsilon x_1), \\ D(x_1) &= \frac{1}{2\lambda^2}\left(\frac{1}{\delta}\sinh\delta x_1 - \frac{\sin\varepsilon x_1}{\varepsilon}\right). \end{aligned}\right\} \tag{9.11}$$

Observe that

$$\left.\begin{aligned} X(x_1) &= X(0)A(x)+X'(0)B(x_1)+X''(0)C(x_1)+X'''(0)D(x), \\ X'(x_1) &= X(0)\delta_\varepsilon^2{}^2D(x_1)+X'(0)A(x_1)+X''(0)C'(x_1)+X'''(0)C(x_1), \\ X''(x_1) &= X(0)\delta_\varepsilon^2{}^2C(x_1)+X'(0)\delta_\varepsilon^2{}^2D(x_1)+X''(0)C''(x_1)+X'''(0)C'(x_1), \\ X'''(x_1) &= X(0)\delta_\varepsilon^2{}^2C'(x_1)+X'(0)\delta_\varepsilon^2{}^2C(x_1)+X''(0)C'''(x_1)+X'''(0)C''(x_1), \end{aligned}\right\} \quad (9.12)$$

where

$$C'(x_1) = B(x_1)+2\beta_m^2D(x_1),\ C''(x_1) = A(x_1)+2\beta_m^2C(x_1),$$

$$C'''(x_1) = \delta_\varepsilon^2{}^2D(x_1)+2\beta_m^2\left[B(x_1)+2\beta_m^2D(x_1)\right].$$

Consider a plate with the clamped edges $x_1 = 0,\ a_1$. In this case we have

$$X(0) = X'(0) = X(a_1) = X'(a_1) = 0, \quad (9.13)$$

and the solution (9.12) has the form

$$X(x_1) = X''(0)C(x_1) + X'''(0)D(x_1). \quad (9.14)$$

Here the two first conditions of the set (9.13) have been used. The remaining conditions lead to the system of equations

$$X(a_1) = X''(0)C(a_1) + X'''(0)D(a_1) = 0$$

$$X'(a_1) = X''(0)C'(a_1) + X'''(0)C(a_1) = 0.$$

This system does not lead to a contradiction if its determinant is equal to zero, i.e. if

$$C'(a_1)D(a_1) - C^2(a_1) = 0.$$

Thus we are led to the relation

$$\left(\delta\tanh\frac{\delta a_1}{2} + \varepsilon\tan\frac{\varepsilon a_1}{2}\right)\left(\delta\coth\frac{\delta a_1}{2} - \varepsilon\cot\frac{\varepsilon a_1}{2}\right) = 0. \quad (9.15)$$

The equation

$$\delta \tanh \frac{\delta a_1}{2} + \varepsilon \tan \frac{\varepsilon a_1}{2} = 0 \tag{9.16}$$

corresponds to the symmetric modes of vibration of the plate, and the equation

$$\delta \operatorname{cotanh} \frac{\delta a_1}{2} - \varepsilon \operatorname{cotan} \frac{\varepsilon a_1}{2} = 0 \tag{9.17}$$

to the antisymmetric modes of vibration.

For a given value of β_m the consecutive value of λ_{nm} can be calculated by means of Equations (9.16) and (9.17); furthermore the consecutive frequencies of vibration are given by the formula

$$\omega_{nm} = c\lambda_{nm}^2 \tag{9.18}$$

The mode of the eigenvibrations is given by Equation $(9.12)_1$, i.e.

$$X(x_1) = X''(0)\left[C(x_1) - \frac{C(a_1)}{D(a_1)}D(x_1)\right]. \tag{9.19}$$

In the particular case of a plate simply supported along the edges $x_1 = 0,\ a_1$ we obtain

$$\sin \varepsilon_n a_1 = 0,\ \varepsilon_n = \frac{n\pi}{a_1},\ n = 1,2,\ldots,\infty.$$

Since

$$\varepsilon_n = \sqrt{\lambda_{nm}^2 - \beta_m^2}$$

we have

$$\lambda_{nm}^2 = \alpha_n^2 + \beta_m^2, \quad \alpha_n = \frac{n\pi}{a_1}.$$

This leads to the result

$$\omega_{nm} = c\lambda_{mn} = (\alpha_n^2+\beta_n^2)\sqrt{\frac{N}{\sigma}}. \tag{9.20}$$

The mode of the eigenvibrations has the form

$$X(x_1) = A\sin \frac{n\pi x_1}{a_1}. \tag{9.21}$$

Consider two different modes of free vibrations of the plate (satisfying the same boundary conditions), namely $W_{ij}(x_1,x_2)$ and $W_{kl}(x_1,x_2)$ with the corresponding eigenvalues λ_{ij} and λ_{kl}.

These satisfy the differential equations

$$\nabla^4 W_{ij}-\lambda_{ij}^4 W_{ij} = 0, \quad \nabla^4 W_{kl}-\lambda_{kl}^4 W_{kl} = 0. \tag{9.22}$$

From these equations we have

$$\iint_A (W_{kl}\nabla^4 W_{ij}-W_{ij}\nabla^4 W_{kl})\mathrm{d}A = (\lambda_{ij}^4-\lambda_{kl}^4)\iint_A W_{ij}W_{kl}\mathrm{d}A. \tag{9.23}$$

But the left side of Equation (9.23) is equal to zero (see (4.2)!), we have

$$(\lambda_{ij}^4-\lambda_{kl}^4)\iint_A W_{ij}W_{kl}\mathrm{d}A = 0. \tag{9.24}$$

Since $\lambda_{ij} \neq \lambda_{kl}$, Equation (9.24) is satisfied only if

$$\iint_A W_{ij}W_{kl}\mathrm{d}A = 0, \quad (i \neq k,\ j \neq l). \tag{9.25}$$

The eigenfunctions of the free vibrations of plates are orthogonal; their coefficients are chosen to satisfy the condition

$$\iint_A W_{ij}^2\mathrm{d}A = 1. \tag{9.26}$$

In the following considerations it will be assumed that the eigenfunctions satisfy both conditions (9.25) and (9.26).

Observe that the orthogonality property holds also for modes of vibration which cannot be written in the form of a product $W_{ij} = X_i(x_1)Y_j(x_2)$.

Consider a plate which undergoes forced and free vibrations.

Assume that at time $t = 0$ the deflection and the velocity of the plate are known, i.e. that

$$w(\underset{\sim}{x},0) = f(\underset{\sim}{x}), \quad \dot{w}(\underset{\sim}{x},0) = g(\underset{\sim}{x}), \quad \underset{\sim}{x} \equiv (x_1,x_2). \tag{9.27}$$

The solution of the differential equation

$$c^2\nabla^4 w + \ddot{w} = \frac{1}{\sigma}q(\underset{\sim}{x},t)$$

takes the form

$$w(\underset{\sim}{x},t) = \int_0^t d\tau \iint_A q(\underset{\sim}{x}',t-\tau)G(\underset{\sim}{x}',\underset{\sim}{x},\tau)dA(\underset{\sim}{x}') \tag{9.28}$$

$$+ \sigma\iint_A \left[g(\underset{\sim}{x}') + f(\underset{\sim}{x}')\frac{\partial}{\partial t}\right]G(\underset{\sim}{x}',\underset{\sim}{x},t)dA(\underset{\sim}{x}').$$

Now, we must solve the differential equation for the Green's function $G(\underset{\sim}{x},\underset{\sim}{x}',t)$:

$$(c^2\nabla^4 + \partial_t^2)G(\underset{\sim}{x},\underset{\sim}{x}',t) = \frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}')\delta(t), \tag{9.29}$$

with the same boundary conditions as the deflection $w(\mathrm{x},t)$ and with homogeneous initial conditions.

Applying the Laplace transform to Equation (9.29) we obtain

$$(c^2\nabla^4+p^2)\overline{G}(\underset{\sim}{x},\underset{\sim}{x}',p) = \frac{1}{\sigma}\delta(\underset{\sim}{x}-\underset{\sim}{x}'). \tag{9.30}$$

Let us now expand the function $\overline{G}$ into a series of the eigenfunctions $W_{ij}(\underset{\sim}{x})$, which satisfy the Equation (9.2) with the same boundary conditions as the functions w and $\overline{G}$. We introduce the notation of finite transform

$$G^*(k,l;x_1',x_2';p) = \int_0^{a_1}\int_0^{a_2} \overline{G}(x_1,x_2;x',x';p)W_{kl}(x_1,x_2)dx_1dx_2, \tag{9.31}$$

$$\overline{G}(x_1,x_2;x_1',x_2';p) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} G^*(k,l;x_1',x_2';p)W_{kl}(x_1,x_2). \tag{9.32}$$

Multiplying Equation (9.30) by $W_{kl}(x_1,x_2)$ and integrating over the whole region of the plate, we obtain

$$(p^2+c^2\lambda_{kl}^4)G^*(k,l;x_1',x_2';p) = \frac{1}{\sigma}W_{kl}(x_1',x_2'). \tag{9.33}$$

Making use of the inverse transform (9.32) and inverting the Laplace transform, we obtain the following expression for the Green's function of the rectangular plate:-

$$G(\underset{\sim}{x},\underset{\sim}{x}',t) = \frac{1}{\sigma}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}W_{kl}(\underset{\sim}{x})W_{kl}(\underset{\sim}{x}')\frac{\sin\omega_{kl}t}{\omega_{kl}}, \quad \omega_{kl} = c\lambda_{kl}^2. \tag{9.34}$$

Consider a particular case of forced aperiodic vibrations. Introducing (9.34) into Equation (9.28) and assuming $g = f = 0$, we have

$$w(\underset{\sim}{x},t) = \frac{1}{\sigma}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}W_{kl}(\underset{\sim}{x})\int_0^{a_1}\int_0^{a_2}W_{kl}(\underset{\sim}{x}')\mathrm{d}A(\underset{\sim}{x}')\int_0^t q(\underset{\sim}{x}',\tau)\frac{1}{\omega_{kl}}\sin\omega_{kl}(t-\tau)\mathrm{d}\tau. \tag{9.35}$$

Suppose that the force $q(\underset{\sim}{x},t)$ moves with the constant velocity V along the line $x_2 = \eta_2$, and hence that

$$q(x_1,x_2,t) = \begin{cases} F(t)\delta(x_1-Vt)\delta(x_2-\eta_2) & \text{for } 0 < Vt < a_1 \\ 0 & \text{for } Vt > a_1. \end{cases}$$

Inserting $q(\underset{\sim}{x},t)$ into Equation (9.35) and taking into account the relation

$$\int_0^{a_1}\int_0^{a_2}\delta(x_1'-Vt)\delta(x_2'-\eta_2)W_{kl}(x_1',x_2')\mathrm{d}x_1'\mathrm{d}x_2' = W_{kl}(Vt,\eta_2)$$

we arrive at the formula

$$w(\underset{\sim}{x},t) = \frac{1}{\sigma}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}W_{kl}(\underset{\sim}{x})\int_0^t F(\tau)W_{kl}(V\tau,\eta_2)\frac{1}{\omega kl}\sin\omega_{kl}(t-\tau)\mathrm{d}\tau. \tag{9.36}$$

If $F(t) = P_0H(t)$, where $H(t)$ is the Heaviside-step function, then

$$w(\underset{\sim}{x},t) = \frac{P_0}{\sigma}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}W_{kl}(\underset{\sim}{x})\int_0^t W_{kl}(V\tau,\eta_2)\frac{1}{\omega_{kl}}\sin\omega_{kl}(t-\tau)d\tau . \tag{9.37}$$

In the case of a plate simply supported along the whole boundary a particularly simple expression for the deflection of the plate is obtained. Since

$$W_{kl}(\underset{\sim}{x}) = \frac{2}{\sqrt{a_1a_2}}\sin\alpha_k x_1 \sin\beta_l x_2$$

$$\alpha_n = \frac{n\pi}{a_1},\ \beta_m = \frac{m\pi}{a_2},\quad \omega_{kl}^2 = (\alpha_k^2 + \beta_l^2)\frac{N}{\sigma},$$

we obtain after carrying out the integration indicated in Equation (9.37)

$$w(\underset{\sim}{x},t) = \frac{4P_0}{\sigma a_1a_2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sin\alpha_k x_1\sin\beta_l x_2\sin\beta_l\eta_2}{\omega_{kl}(\alpha_k^2V^2-\omega_{kl}^2)}\left[\alpha_k V\sin\omega_{kl}t-\omega_{kl}\sin\alpha_k Vt\right]. \tag{9.38}$$

$$0 < Vt < a.$$

If $V \to 0$, and $\sin\alpha_k Vt \to \sin\alpha_k\eta_1$ Equation (9.38) gives the statical deflection of the plate produced by the force P_0 located at the point (η_1,η_2)

$$w(\underset{\sim}{x},t) = \frac{4P_0}{a_1a_2N}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sin\alpha_k x_1\sin\alpha_k\eta_1\sin\beta_l x_2\sin\beta_l\eta_2}{(\alpha_k^2+\beta_l^2)^2}. \tag{9.39}$$

Having found the deflection surface, we can calculate the stresses occurring in the plate with the aid of Equations (3.34).

10. Transverse vibrations of rods and plates resting on an elastic foundation

The differential equation of the transverse vibrations of a rod resting on an elastic foundation has the form

$$c^2\frac{\partial^4 w}{\partial x^4} + \ddot{w} = \frac{1}{\sigma}(q-r), \quad c^2 = \frac{EI}{\sigma}, \quad \sigma = \rho A. \tag{10.1}$$

Assuming a linear relation between the resistance r of the foundation and the deflection (Winkler's foundation), we have

$$r(x,t) = kw(x,t) \tag{10.2}$$

where k is the foundation modulus. Thus Equation (10.1) takes the form

$$c^2\frac{\partial^4 w}{\partial x^4} + \ddot{w} + \kappa^2 w = \frac{1}{\sigma}q(x,t), \qquad \kappa^2 = {}^{k}/_{\sigma}. \tag{10.3}$$

Equation (10.3) is only an approximation to real conditions. It is valid only for small deflections. The assumption on $r(x,t)$ states that the resistance $r(x,t)$ produces a deflection only in the section x while in fact $w(x,t)$ depends on the resistance at all points of the rod. We assume also that during the deformation the rod is in contact with the foundation over the whole length. It is therefore clear that Equation (10.3) only approximately describes the phenomenon of vibration.

Consider now an infinite rod resting in an elastic foundation, which at time $t = 0$ is subject to the instantaneous loading $q(x,t) = P_0\delta(x)\delta(t)$.

To Equation (10.3) we first apply the Laplace transform with respect to time and then the Fourier cosine transform. Inverting the Fourier cosine transform, we have

$$\begin{aligned}\overline{w}(x,p) &= \frac{P_0}{\pi EI}\int_0^\infty \frac{\cos\alpha x\, d\alpha}{\alpha^4+4\mu^4} \\ &= \frac{P_0 e^{-\mu k}}{8\mu^3 EI}(\cos\mu x+\sin\mu x), \ \mu = \left(\frac{k+p^2\sigma}{4EI}\right)^{\frac{1}{4}}\end{aligned} \tag{10.4}$$

The inversion of the Laplace transform involves serious difficulties. In the particular case $w(0,t)$ we obtain

$$w(0,t) = \frac{P_0\Gamma(\frac{3}{4})}{2\sqrt{\pi}(4EI)^{\frac{1}{4}}(2k)^{3/2}}(t\,\frac{k}{\sigma})^{\frac{1}{4}}J_{\frac{1}{4}}(t\,\frac{k}{\sigma}), \tag{10.5}$$

where $J_{\frac{1}{4}}(z)$ is the Bessel function of first kind of order $\frac{1}{4}$. For the static case, we have

$$w(x) = \frac{P_0}{8EI\eta^3}e^{-\eta x}(\cos\eta x+\sin\eta x), \quad \eta = (\frac{k}{4EI})^{\frac{1}{4}}, \; x > 0. \tag{10.6}$$

The differential equation of the transverse vibrations of a plate resting on an elastic foundation has the form

$$c^2\nabla^4 w+\ddot{w}+\kappa^2 w = \frac{1}{\sigma}q(x,t), \quad c^2 = \frac{N}{\sigma}, \quad \sigma = \rho h, \quad \kappa^2 = {}^{k}/_{\sigma}. \tag{10.7}$$

Consider the infinite plate resting on an elastic foundation, subject to the static loading. The load $q(r) = \dfrac{P_0\delta(r)}{2\pi r}$ is applied to the infinite plate. The problem of determining the deflection of the plate is an axisymmetric one.

$$c^2\nabla_r^4 w+\eta^4 w = \frac{P_0}{\sigma}\,\frac{\delta(r)}{2\pi r}, \qquad \nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \quad \eta = (\frac{k}{\sigma})^{\frac{1}{4}}. \tag{10.8}$$

Multiplying Equation (10.8) by $rJ_0(\alpha r)$ and integrating with respect to r in the interval $<0,\infty>$, we transform Equation (10.8) to the form

$$(c^2\alpha^4+\eta^4)\bar{w}(\alpha) = \frac{P_0}{2\pi\sigma}, \qquad \bar{w}(\alpha) = \int_0^\infty w(r)rJ_0(\alpha r)dr. \tag{10.9}$$

Applying the inverse Hankel transform, we have

$$w(r) = \frac{P_0}{2\pi\sigma c^2}\int_0^\infty \frac{\alpha J_0(\alpha r)d\alpha}{\alpha^4+\eta^4/c^2}, \tag{10.10}$$

or

$$w(r) = -\frac{P_0}{2\pi N} kei_0\left[r\left(\frac{k}{N}\right)^{\frac{1}{4}}\right], \qquad r = (x_1^2 + x_2^2)^{\frac{1}{2}}. \tag{10.11}$$

There $kei_0(z)$ is the modified Kelvin function. Assuming $P_0 = 1$ and moving the concentrated force to the point x', we obtain

$$G(\underset{\sim}{x},\underset{\sim}{x}') = -\frac{1}{2\pi N} kei_0\left[r\left(\frac{k}{N}\right)^{\frac{1}{4}}\right], \qquad r = \left[(x_1-x_1')^2+(x_2-x_2')^2\right]^{\frac{1}{2}}. \tag{10.12}$$

Consider a plate-strip resting on an elastic foundation, simply supported at the edges $x_1 = 0, a_1$ and subjected to the action of a concentrated force $q(x,t) = P_0\delta(x_1-x_1')\delta(x_2)$.

We have to solve the equation

$$\nabla^4 w+\mu^4 w = \frac{P_0}{N}\delta(x_1-x_1')\delta(x_2), \qquad \mu^4 = \frac{k}{\sigma c^2} = \frac{k}{N}. \tag{10.13}$$

Using the finite sine transform and the integral cosine transform, we obtain

$$w(\underset{\sim}{x}) = \frac{2P_0}{\pi N a_1}\sum_{n=1}^{\infty}\sin\alpha_n x_1'\sin\alpha_n x_1\int_0^{\infty}\frac{\cos\beta x_2 d\beta}{(\alpha_n{}^2+\beta^2)^2+\mu^4}, \quad \alpha_n = \frac{n\pi}{a_1} \tag{10.14}$$

The solution of the infinite integral has the form

$$\int_0^{\infty}\frac{\cos\beta x_2 d\beta}{(\beta^2+\gamma_n{}^2)(\beta^2+\delta_n{}^2)} = \frac{\pi}{2}\frac{1}{\delta_n{}^2-\gamma_n{}^2}\left(\frac{e^{-\gamma_n x_2}}{\gamma_n} - \frac{e^{-\delta_n x_2}}{\delta_n}\right), \tag{10.15}$$

where

$$\gamma_n{}^2 = \alpha_n{}^2+i\mu^2, \quad \delta_n{}^2 = \alpha_n{}^2-i\mu^2, \quad \delta_n{}^2-\gamma_n{}^2 = -2i\mu^2.$$

We must take the real part of the integral (10.15). In the particular case $k = 0$, we have to do with the integral

$$\int_0^\infty \frac{\cos\beta x_2 d\beta}{(\alpha_n^2+\beta^2)^2} = \frac{\pi}{4\alpha_n^3}(1+\alpha_n x_2)e^{-\alpha_n x_2}, \quad x_2 > 0. \tag{10.16}$$

The deflection $w(\underset{\sim}{x})$ takes the form

$$w(\underset{\sim}{x}) = \frac{P_0}{2a_1 N}\sum_{n=1}^{\infty}\frac{(1+\alpha_n x_2)}{\alpha_n^3}e^{-\alpha_n x_2}\sin\alpha_n x_1'\sin\alpha_n x_1, \; x_2 > 0. \tag{10.17}$$

From (10.17) we obtain the Green's function

$$G(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{2a_1 N}\sum_{n=1}^{\infty}\frac{1}{\alpha_n}(1+\alpha_n(x_2-x_2'))e^{-\alpha_n(x_2-x_2')}\sin\alpha_n x_1'\sin\alpha_n x_1, \; x_2-x_2'>0. \tag{10.18}$$

c

Applying the operator ∇^2 to the function $G(\underset{\sim}{x},\underset{\sim}{x}')$, we have

$$\nabla^2 G(\underset{\sim}{x},\underset{\sim}{x}') = \phi(\underset{\sim}{x},\underset{\sim}{x}') = -\frac{1}{a_1 N}\sum_{n=1}^{\infty}\frac{\exp\left[-\alpha_n(x_2-x_2')\right]}{\alpha_n}\sin\alpha_n x_1\sin\alpha_n x_1' \tag{10.19}$$

or

$$\phi(\underset{\sim}{x},\underset{\sim}{x}') = \frac{1}{4\pi N}ln\frac{\cosh\frac{\pi}{a}(x_2-x_2')-\cos\frac{\pi}{a}(x_1-x_1')}{\cosh\frac{\pi}{a}(x_2-x_2')-\cos\frac{\pi}{a}(x_1+x_1')}. \tag{10.20}$$

Differentiating the function $\phi(\underset{\sim}{x},\underset{\sim}{x}')$, we obtain

$$\left.\begin{aligned} 2N\frac{\partial^2 G}{\partial x_1^2} &= \phi - (x_2-x_2')\frac{\partial\phi}{\partial x_2}, \\ 2N\frac{\partial^2 G}{\partial x_2^2} &= \phi + (x_2-x_2')\frac{\partial\phi}{\partial x_2}, \\ 2N\frac{\partial^2 G}{\partial x_1\partial x_2} &= (x_2-x_2')\frac{\partial\phi}{\partial x_1}. \end{aligned}\right\} \tag{10.21}$$

Now we can determine the bending and twisting moment in the closed form

$$\left.\begin{aligned} M_{11}(G) &= -N(G_{,11}+\nu G_{,22}) = -\frac{1+\nu}{2}\phi + \frac{1-\nu}{2}(x_2-x_2')\frac{\partial\phi}{\partial x_2}, \\ M_{22}(G) &= -N(G_{,22}+\nu G_{,11}) = -\frac{1+\nu}{2}\phi - \frac{1-\nu}{2}(x_2-x_2')\frac{\partial\phi}{\partial x_2}, \\ M_{12}(G) &= -N(1-\nu)G_{,12} = -\frac{1-\nu}{2}\frac{\partial\phi}{\partial x_1}. \end{aligned}\right\} \quad (10.22)$$

The form of the shear forces is the following

$$\begin{aligned} Q_1(G) &= -N\frac{\partial\nabla^2 G}{\partial x_1} = -\frac{\partial\phi}{\partial x_1}, \\ Q_2(G) &= -N\frac{\partial\nabla^2 G}{\partial x_2} = -\frac{\partial\phi}{\partial x_2}. \end{aligned} \quad (10.23)$$

11. Free vibrations of an infinite rod and of an infinite plate.

Consider the integral expression (4.13) for the special case: $q = 0$, $g = 0$.

$$w(x,t) = \sigma\int_{-\infty}^{\infty} f(x')\frac{\partial G(x',x,t)}{\partial t}dx'. \quad (11.1)$$

We have to solve the differential equation for the Green's function

$$\left(c^2\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial t^2}\right)G(x,x',t) = \frac{1}{\sigma}\delta(x-x')\delta(t),\ c^2 = \frac{EI}{\sigma},\ \sigma = \rho A. \quad (11.2)$$

$$-\infty < x < \infty,\ t > 0.$$

Let us perform over the differential equation (11.2) the Laplace transform and further the Fourier exponential transform.

Inverting the Fourier transform, we have

$$p\bar{G}(x,x',p) = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\frac{p}{c^2\xi^4+p^2}e^{-i\xi(x-x')}d\xi. \tag{11.3}$$

Now we invert the Laplace transform. Since

$$\mathcal{L}^{-1}\left(\frac{p}{c^2\xi^4+p^2}\right) = \cos\xi^2 ct$$

we obtain

$$\frac{\partial G}{\partial t} = \frac{1}{2\pi\sigma}\int_{-\infty}^{\infty}\cos\xi^2 ct e^{-i\xi(x-x')}d\xi. \tag{11.4}$$

The following relation will be employed

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos\xi^2 ct e^{-i\xi(x-x')}d\xi = \frac{1}{2\sqrt{ct}}\left[\cos\frac{(x-x')^2}{4ct} + \sin\frac{(x-x')^2}{4ct}\right]. \tag{11.5}$$

Introducing (11.5) into the integral expression (11.1), we obtain

$$w(x,t) = \frac{1}{2\sqrt{2\pi ct}}\int_{-\infty}^{\infty}f(x')\left[\cos\frac{(x-x')^2}{4ct} + \sin\frac{(x-x')^2}{4ct}\right]dx' \tag{11.6}$$

or

$$w(x,t) = \frac{1}{2\sqrt{2\pi ct}}\int_{-\infty}^{\infty}f(x-x')\left[\cos\frac{x'^2}{4ct} + \sin\frac{x'^2}{4ct}\right]dx'. \tag{11.7}$$

In the particular case

$$f(x) = f_0\exp\left(\frac{-x^2}{4a^2}\right)$$

i.e. when the initial curve has a prescribed form, we obtain from (11.7)

$$w(x,t) = \frac{f_0}{\sqrt{1+c^2t^2/a^4}}\exp\frac{-x^2a^2}{4(a^4+c^2t^2)}\cos\left[\frac{ctx^2}{4(a^4+c^2t^2)} - \tfrac{1}{2}\tan^{-1}\left(\frac{ct}{a^2}\right)\right] \tag{11.8}$$

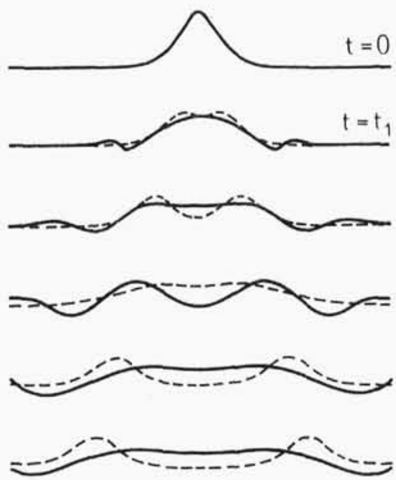

Fig. 11.1

Figure 11.1 represents the graphs of the function $w(x,t)$ for consecutive values of parameter t. For the sake of comparison the dotted line represents the propagation of a transverse elastic wave in a string, for the same values of t. In the latter case we have two "crests" propagated in the opposite directions, for a rod, however, such a division does not occur.

Let us discuss the problem of free vibrations of an infinite plate and confine our considerations to axisymmetric modes of vibrations. Assume that the initial conditions depend on the variable r only, and are independent of the angle θ. The equation of the transverse vibration of the plate can be written in cylindrical coordinates, namely

$$c^2\nabla_r^4 w(r,t) + w(r,t) = 0. \tag{11.9}$$

The equation (11.1) takes for the axisymmetric form of vibration the form

$$w(r,t) = \sigma\int_0^\infty f(r')\frac{\partial G(r',r,t)}{\partial t}dr'. \tag{11.10}$$

We have to solve the differential equation for the Green's function

$$(c^2\nabla_r^2 + \partial_t^2)G(r,r',t) = \frac{1}{\sigma}\delta(r-r')\delta(t). \tag{11.11}$$

Let us perform over the differential equation (11.11) the Laplace transform and further the Hankel transformation. After inverting, we obtain

$$\frac{\partial G}{\partial t} = \frac{1}{\sigma}\int_0^\infty r'J_0(\alpha r')\alpha J_0(\alpha r)\cos(\alpha^2 ct)d\alpha. \tag{11.12}$$

Introducing (11.12) into the integral expression (11.10) we obtain

$$w(r,t) = \int_0^\infty r'f(r')dr'\int_0^\infty \alpha J_0(\alpha r')J_0(\alpha r)\cos(c\alpha^2 t)d\alpha. \tag{11.13}$$

Consider the Weber integral

$$\int_0^\infty \alpha J_0(\alpha r')J_0(\alpha r)e^{-\nu\alpha^2}d\alpha = \frac{1}{2\nu}\exp(-\frac{r'^2+r^2}{4ct})I_0(\frac{rr'}{2\nu}), I_0(z) = J_0(iz). \tag{11.14}$$

Inserting $\nu = -ict$ into Equation (11.14) and taking the real part of the integral, we have

$$\int_0^\infty \alpha J_0(\alpha r')J_0(\alpha r)\cos(\alpha^2 ct)d\alpha = \frac{1}{2ct}J_0(\frac{rr'}{2ct})\sin(\frac{r^2+r'^2}{4ct}).$$

This procedure leads to the following final form of the formula for the free vibration of the plate

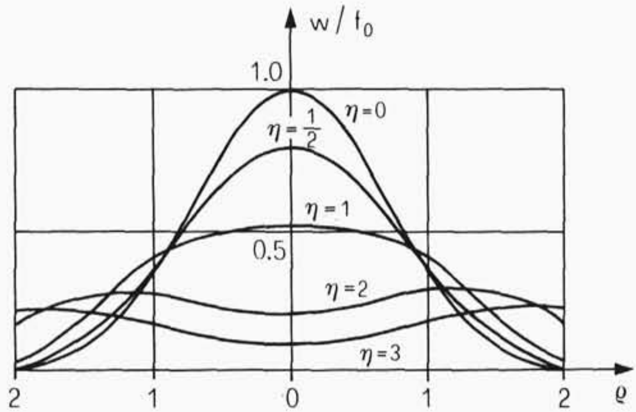

Fig. 11.2

$$w(r,t) = \frac{1}{2ct}\int_0^{\infty} r'f(r')J_0\left(\frac{rr'}{2ct}\right)\sin\left(\frac{r^2+r'^2}{4ct}\right)dr'. \tag{11.15}$$

Let the initial deflection of the plate equal

$$w(r,0) = f(r) = f_0\exp\left(-\frac{r^2}{a^2}\right).$$

From the Equation (11.15) we have

$$w(r,t) = \frac{f_0a^2}{2}\int_0^{\infty} \alpha\exp\left(-\frac{\alpha^2}{4a^2}\right)J_0(\alpha r)\cos(c\alpha^2t)d\alpha. \tag{11.16}$$

This leads to the formula [93]

$$w(r,t) = \frac{f_0}{1+\eta^2}\exp\left(-\frac{\rho^2}{1+\eta^2}\right)\left[\cos\frac{\rho\eta^2}{1+\eta^2} + \eta\sin\frac{\rho\eta^2}{1+\eta^2}\right], \tag{11.17}$$

where $\eta = 4ct/a^2$; $\rho = r/a$.

The deflection w for several values of the parameter η is shown in Figure 11.2.

12. Transverse vibrations of viscoelastic rods and plates.

In formulating the stress-deformation relation of a viscoelastic body, it is convenient to represent them in a form analogous to that of the perfectly elastic body. In the latter the Hooke law has the form

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}, \quad i,j = 1,2,3. \tag{12.1}$$

The system of Equations (12.1) can take a different form, corresponding to the representation of the deformation as a sum of volume and shear deformations. Subtracting from (12.1) the quantity $\frac{1}{3}\delta_{ij}\varepsilon_{kk}$, we have

$$\sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon_{kk} - \frac{1}{3}\sigma_{kk})\delta_{ij}. \tag{12.2}$$

Contracting in (12.1), we obtain

$$\sigma_{kk} = (3\lambda + 2\mu)\varepsilon_{kk}. \tag{12.3}$$

In view of (12.2) the system of Equations (12.1) - (12.3) can be replaced by the system of equations

$$s_{ij} = 2\mu e_{ij}, \tag{12.4}$$

$$s = 3Ke, \tag{12.5}$$

where

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}, \; e_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}, \; \sigma_{kk} = s, \; \varepsilon_{kk} = e, \; K = \lambda + \frac{2}{3}\mu. \tag{12.6}$$

s_{ij} is called the stress deviator and e_{ij} the deformation deviator. The stresses s_{ij} produce a change in shape only, this fact being expressed by formula (12.4), while the mean normal stresses produce a change in volume. The equations contain two constants, the shear modulus $\mu = G$ and the bulk modulus K. We now proceed to a viscoelastic body. Assuming that in all round tension (compression) the body behaves as perfectly elastic, Equation (12.5) remains unaltered. Relations (12.4) are generalized by adding to the right hand side a term representing the Newtonian viscosity, i.e. the term $2\eta\dot{e}_{ij}$. Thus

$$s_{ij} = 2\mu(1 + t_r\partial_t)e_{ij}, \quad s = 3Ke, \quad t_r = \eta/_\mu. \tag{12.7}$$

Equation (12.7) represents the Kelvin model. The quantity $t_r = \eta/_\mu$ is called the retardation time.

The relation

$$\dot{s}_{ij} + \frac{s_{ij}}{t_r} = 2\mu e_{ij}, \qquad s = 3Ke, \quad t_r = \eta/_\mu, \tag{12.8}$$

occurs for the viscoleastic Maxwell body.

General stress-deformation relations for linear viscoelastic bodies can be represented in a form analogous to (12.4) and (12.5), viz.

$$P_1(D)s_{ij}(\underset{\sim}{x},t) = P_2(D)e_{ij}(\underset{\sim}{x},t), \tag{12.9}$$

$$P_3(D)s(\underset{\sim}{x},t) = P_4(D)e(\underset{\sim}{x},t), \quad \underset{\sim}{x} = (x_1,x_2,x_3), \tag{12.10}$$

where

$$P_i(D) = \sum_{n=0}^{N_i} a_i^{(n)}D^n, \; a_i^{(N_i)} \neq 0, \; i = 1,2,3,4,$$

are differential operators and $D^n = \dfrac{\partial^n}{\partial t^n}$ denotes the n time derivative, $a_i^{(n)}$ are constant coefficients.

Assume that the viscoelastic body is in natural state for $t < 0$, i.e. there are no stresses and deformations, and the loading is applied at the instant $t = 0$. Under these assumptions we may apply to Equations (12.9), (12.10) the one-sided Laplace transform.

$$\bar{s}_{ij}(\underset{\sim}{x},p) = 2\bar{\mu}(p)\bar{e}_{ij}(\underset{\sim}{x},p) \tag{12.11}$$

$$\bar{s}(\underset{\sim}{x},p) = 3\bar{K}(p)\bar{e}(\underset{\sim}{x},p), \tag{12.12}$$

where

$$\bar{\mu}(p) = \frac{P_2(p)}{2P_1(p)}, \quad \bar{K}(p) = \frac{1}{3}\frac{P_4(p)}{P_3(p)} .$$

Observe that Equations (12.1) and (12.2) are of analogous structure to Equations (12.4) and (12.5) for the perfectly elastic body. However, in the latter, the quantities μ, K appearing in Equations (12.4) and (12.5) are constants, while in the case of viscoelasticity we are dealing with functions of the parameter p. Equations (12.11) and (12.12) can be solved with respect to the stresses. Thus, we obtain the relations

$$\bar{\sigma}_{ij}(\underset{\sim}{x},p) = 2\bar{\mu}(p)\bar{\varepsilon}_{ij}(\underset{\sim}{x},p) + \bar{\lambda}(p)\delta_{ij}\bar{e}(\underset{\sim}{x},p), \tag{12.13}$$

where

$$\bar{\lambda}(p) = \frac{P_1(p)P_2(p)-P_2(p)P_3(p)}{3P_1(p)P_3(p)}.$$

We have already indicated the analogy between formulae (12.13) and (12.1). The analogies may be employed in constructing the stress-deformation relations for plane states of stress, one dimensional state of stress, etc.

In the plane state of stress, we have for the elastic body

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2}\left[(1-\nu)\varepsilon_{\alpha\beta}+\nu e\delta_{\alpha\beta}\right], \; e = \varepsilon_{11} + \varepsilon_{22}, \; \alpha,\beta = 1,2 \tag{12.14}$$

where

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}, \quad K = \lambda + \frac{2}{3}\mu.$$

For the plane state of stress in a viscoelastic body

$$\bar{\sigma}_{\alpha\beta} = \frac{\bar{E}}{1-\bar{\nu}^2}\left[\bar{\varepsilon}_{ij}(1-\bar{\nu})+\bar{\nu}\delta_{ij}\bar{e}\right], \quad \alpha,\ \beta = 1,2 \tag{12.15}$$

the quantities $\bar{E}$, $\bar{\nu}$ being expressed in terms of $\bar{\mu}(p)$, $\bar{\lambda}(p)$ by the formulae

$$\bar{E} = \frac{\bar{\mu}(3\bar{\lambda}+2\bar{\mu})}{\bar{\lambda}+\bar{\mu}}, \quad \bar{\nu} = \frac{\bar{\lambda}}{2(\bar{\lambda}+\bar{\mu})}, \quad \bar{K} = \bar{\lambda} + \frac{2}{3}\bar{\mu}. \tag{12.16}$$

The above elastic-viscoelastic analogy was announced by Alfrey and Lee. To solve a dynamic problem of viscoelasticity we can use the corresponding solutions of the perfectly elastic problem, replacing in the latter λ, μ by $\bar{\lambda}$, $\bar{\mu}$ and inverting the Laplace transform.

Consider now an example of transverse vibrations of rods of viscoelastic material. The differential equation of the transverse vibration of a rod of perfectly elastic material (after applying the Laplace transform) has the form

$$c^2\frac{d^4\bar{w}}{dx^4} + p^2\bar{w} = \frac{1}{\sigma}\bar{q}(x,p) + pf(x) + g(x), \tag{12.17}$$

where

$$\bar{w}(x,p) = \int_0^\infty e^{-pt}w(x,t)dt, \quad c = \sqrt{\frac{EI}{\rho h}}.$$

For a viscoelastic body, c should be replaced by $\bar{c}(p)$, the latter being a function of p, the parameter of the Laplace transform. The transformed equation of vibrations of a viscoelastic rod has therefore the form

$$\bar{c}^2(p)\frac{d^4\bar{w}}{dx^4} + p^2\bar{w} = \frac{1}{\sigma}\bar{q}(x,p) + pf(x) + g(x), \quad \bar{c}^2(p) = \frac{I\bar{E}(p)}{\sigma}. \tag{12.18}$$

For the viscoelastic body

$$\overline{E}(p) = \frac{\overline{\mu}(p)(3\overline{\lambda}(p)+2\overline{\mu}(p))}{\overline{\lambda}(p)+\overline{\mu}(p)}.$$

Suppose that $\overline{w}$, $f(x)$, $g(x)$can be expanded into a series of eigenfunctions of the rod of perfectly elastic material, with the same conditions of support as the rod under consideration:

$$\overline{w}(x,p) = \sum_{n=1}^{\infty} w^*(n,p)W_n(x),\ f(x) = \sum_{n=1}^{\infty} f(n)W_n(x), \tag{12.19}$$

$$g(x) = \sum_{n=1}^{\infty} g(n)W_n(x),\ \overline{q}(x,p) = \sum_{n=1}^{\infty} q^*(n,p)W_n(x).$$

The functions $W_n(x)$ are orthogonal and normalized, and they satisfy the equation

$$\frac{d^4W_n}{dx^4} - \lambda_n^4 W_n = 0,\quad \lambda_n^4 = \omega_n^2/c_0^2,\quad c_0^2 = \frac{EI}{\sigma}. \tag{12.20}$$

Introducing (12.19) into (12.18) we obtain

$$(\overline{c}^2(p)\lambda_n^4+p^2)w^*(n,p) = \frac{1}{\sigma}q^*(n,p) + pf(n) + g(n). \tag{12.21}$$

Introducing (12.21) into (12.19), we have

$$\overline{w}(x,p) = \frac{1}{\sigma}\sum_{n=1}^{\infty}\frac{q^*(n,p)+\sigma(pf(n)+g(n))}{\overline{c}^2(p)\lambda_n^4 + p^2}W_n(x), \tag{12.22}$$

or

$$\overline{w}(x,p) = \frac{1}{\sigma}\sum_{n=1}^{\infty}\frac{W_n(x)}{\overline{c}^2(p)\lambda_n^4+p^2}\int_0^l \overline{q}(x',p)W_n(x')dx' \tag{12.23}$$

$$+\sum_{n=1}^{\infty}\frac{W_n(x)}{\overline{c}^2(p)\lambda_n^4+p^2}\left[p\int_0^l f(x')W_n(x')dx' + \int_0^l g(x')W_n(x')dx'\right]$$

We have now to carry out the required integration and then to invert in (12.23) the Laplace transform. The difficulty of carrying out the latter operation is due to the complicated form of the function $\bar{c}^2(p)$.

Taking into account that, in accordance with relations (12.12) and (12.13)

$$\bar{\lambda}(p) = \frac{P_1(p)P_4(p)-P_2(p)P_3(p)}{3P_1(p)P_3(p)}, \quad \bar{\mu}(p) = \frac{P_2(p)}{2P_1(p)},$$

we obtain

$$\bar{c}^2(p) = \frac{E(p)I}{\sigma} = \frac{I}{\sigma}\,\frac{3P_2(p)P_4(p)}{2P_1(p)P_4(p)+P_2(p)P_3(p)}. \tag{12.24}$$

In the particular case of a Kelvin body, we have

$$P_1(p) = 1,\; P_2(p) = 2G(1+t_r p),\; P_3(p) = p,\;\; P_4(p) = 3Kp.$$

Hence

$$\bar{c}^2(p) = \frac{I}{\sigma}\,\frac{9G(1+t_r p)}{3 + \frac{G}{K}(1+t_r p)}. \tag{12.25}$$

Introducing (12.24) into (12.23) and inverting the Laplace transform, we arrive at the required function $w(x,t)$. A considerable simplification of the expression $\bar{c}^2(p)$ follows if we assume that the material is incompressible ($K \to \infty$, $\nu = \frac{1}{2}$). Then, setting $P_4(p) \to \infty$ in formula (12.24), we have

$$\bar{c}^2(p) = \frac{3}{2}\,\frac{I}{\sigma}\,\frac{P_2(p)}{P_1(p)} = \frac{3GI}{\sigma}(1+t_r p). \tag{12.26}$$

Introducing (12.26) into (12.23) and inverting the Laplace transform, we obtain the solution of our problem:

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\left\{\int_0^l f(x')W_n(x')\mathrm{d}x' \frac{1}{\sqrt{1-\beta_n^2}} e^{-\beta_n\omega_n t}\sin\left[\omega_n t\sqrt{1-\beta_n^2} + \phi_n\right] + \right.$$

$$\left. + \frac{1}{\omega_n\sqrt{1-\beta_n^2}} e^{-\beta_n\omega_n t}\sin(\omega_n t\sqrt{1-\beta_n^2})\int_0^l g(x')W_n(x')\mathrm{d}x'\right\} \qquad (12.27)$$

$$+ \frac{1}{\sigma}\sum_{n=1}^{\infty}\frac{W_n(x)}{\omega_n(1-\beta_n^2)}\int_0^l W_n(x')\mathrm{d}x'\int_0^t q(x',\tau)e^{-\beta_n\omega_n(t-\tau)}\sin\left[\omega_n(t-\tau)\sqrt{1-\beta_n^2}\right]\mathrm{d}\tau,$$

where

$$\beta_n = \frac{\omega_n t_r}{2}, \qquad \phi_n = \tan^{-1}\left[\frac{\sqrt{1-\beta_n^2}}{-\beta_n}\right].$$